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OVERSAMPLED COMPLEX-MODULATED CAUSAL IIR FILTER BANKS FOR FLEXIBLE FREQUENCY-BAND REALLOCATION NETWORKS

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ABSTRACT

This paper introduces a class of oversampled complex-modulated causal IIR filter banks for flexible frequency-band reallocation networks. In the simplest case, they have near perfect magnitude reconstruction (NPMR), but by adding a phase equalizer they can achieve near-PR.

1. INTRODUCTION

The European Space Agency (ESA) outlines three major "standard architectures" for future satellite-based broadband systems [1]. Two of these are the distributed access network and professional user network which are to provide high-capacity point-to-point and multicast services for ubiquitous Internet access. The satellites are to communicate with user units via multiple spot beams. In order to use the limited available frequency spectrum efficiently, the satellite on-board signal processing must support frequency-band reusage among the beams and also flexibility in bandwidth and transmission power allocated to each user. Further, dynamic frequency allocation is desired for covering different service types requiring different data rates and bandwidths. An important issue in the next-generation satellite-based communication system is therefore the on-board reallocation of information. In technical terms, this calls for digital flexible frequency-band reallocation (FBR) networks which thus are critical components.

In a straightforward realization of the flexible FBR network, the filter banks (FBs) need to incorporate variable filters which are costly to implement. Therefore, the authors of [2] propose a solution with fixed FBs as shown in Fig. 1 [1]. A *channel switch* and some complex multipliers μ_{kr} (for phase rotation compensation), together with properly chosen FBs, are used to achieve the necessary flexibility. This solution reduces the arithmetic complexity significantly and outperforms the previously existing networks when flexibility and low complexity are considered simultaneously [2]. The channel switch redirects its inputs $x_r(n)$ at the different positions to its new output positions. A variable *channel combiner* is also needed after the synthesis bank to combine the contents in the different granularity bands into each output $y_r(n)$. The FBR network is based on a class of variable oversampled complex-modulated N-channel FBs which makes use of decimation and interpolation by M. The input and output signals contain an adjustable number of user subbands, q with $1 \leq q \leq Q$. The input and output subbands are denoted $x_r(n)$ and $y_r(n)$, respectively, with $r = 0, 1, \ldots, q-1$. The constant Q is the number of granularity bands and is related to M as $M = BQ$ with the integer $B \geq 1$. In the final implementation, M and N are fixed, whereas q is on-line variable. Further, $N = AQ$, which means that $N = (A/B)M$ with A being an integer greater than B. For more details, see [2].

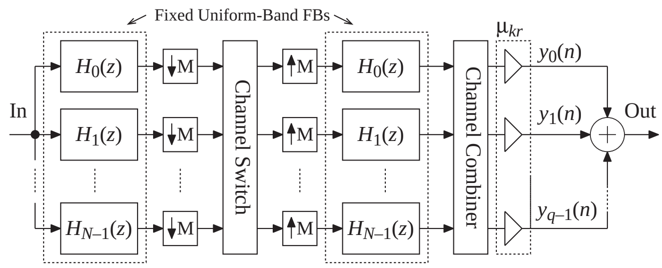

Figure 1: Proposed flexible FBR network with fixed analysis and synthesis FBs. The variable parts are the channel switch, μ_{kr}, and the channel combiner.

In this paper, we introduce a class of complex-modulated causal IIR FBs for the flexible FBR networks mentioned above. Complex-modulated FBs are known to be very efficient since each of the analysis and synthesis parts can be implemented with the aid of one prototype filter alone and an IDFT (DFT) [3]. FIR filters (used in [2]) offer several good properties, like guaranteed stability and exact linear phase. However, in general they need a higher filter order to fulfill a given specification compared to IIR filters; especially when the transition band is narrow. In this paper, we therefore introduce IIR prototype filters instead of FIR ones, the latter being used in most publications on modulated FBs (see e.g. [3] and [4]). This allows us to lower even further the implementation complexity as well as the design complexity of the proposed flexible FBR network.

The FBs do not fulfill perfect reconstruction (PR). However near-PR can be achieved. This means that the FB introduces small magnitude and phase errors. Further, if the phase response is not approximately linear, we say that it is a near perfect magnitude reconstruction (NPMR) FB. In the simplest case, the introduced FBs have NPMR, and by introducing a phase equalizer, the NPR FBs are obtained. Using NPR or NPMR FBs, instead of PR ones, is usually advantageous since the complexity thereby can be reduced [5], [6]. The proposed solution offers additional flexibility, since the optional equalizer can be implemented either in the satellite or at the receiver on earth. Finally, a general design procedure is proposed which can be used to design the FBs to approximate general specifications as close as desired.

[1] A practical multicast systems requires a multiple-input multiple-output (MIMO) network. However, properly designed FBs for single-input single-output (SISO) networks can be used in MIMO networks as well. Therefore only SISO networks are considered in this paper.

2. PROPOSED FILTER BANK CLASS

This section treats filter transfer functions, distortion function and aliasing of the proposed FB class.

2.1 Filter transfer functions

Let the transfer function of the lowpass prototype filter be given as

$$P(z) = \frac{A(z)}{C(z^N)} \tag{1}$$

where N is even[2]. The filters $A(z)$ and $C(z)$ are of order N_A and N_C, respectively, and given by

$$A(z) = \sum_{n=0}^{N_A} a(n)z^{-n} \qquad C(z) = \sum_{n=0}^{N_C} c(n)z^{-n}.$$

Further, $A(z)$ is a linear-phase FIR filter with symmetric impulse response $a(n) = a(N_A - n)$. The analysis filters $H_k(z)$ are obtained by modulation of the prototype filter $P(z)$ according to

$$H_k(z) = \beta_k P(zW_N^{(k+\alpha)}), \quad k = 0, 1, \ldots, N-1 \tag{2}$$

where $W_N = e^{-j2\pi/N}$ and $\beta_k = W_N^{(k+\alpha)N_A/2}$. The constant α is real-valued and arbitrary. Using (1) and (2), $H_k(z)$ can be rewritten as

$$H_k(z) = \frac{A_k(z)}{C(W_N^{\alpha N} z^N)}, \quad A_k(z) = \beta_k A(zW_N^{(k+\alpha)}).$$

All the filters $H_k(z)$ thus share the same denominator polynomial which leads to few optimization parameters and an efficient implementation at the end.

2.2 Distortion function and aliasing

In [2], general formulas for the distortion function are given. These formulas depend on the number of input subbands, the length of each subband and how each subband is reallocated in the network (the channel switch). It is also shown that all μ_{kr} become equal to unity by introducing some additional delay to the system.

As to the filter design, this paper's focus is on the fixed FBs and their properties as it has been shown in [2] that a properly designed FB for the special case with only one input subband (covering the whole frequency range) is enough to ensure that the system also works for all possible combinations of input subbands and reallocation schemes [3]. The price to pay is a slight overdesign. If we add the restriction that A/B must be an integer, this fact will hold also for the proposed FB class. Thus, when designing FBs in this paper we study only the special case with $q = 1$ $(r = 0)$ and we restrict A/B to be an integer. With these choices, the synthesis filters $G_k(z)$ described in [2] become identical to the analysis filters (all $\mu_{kr} = 1$) and the distortion function can be written as

$$V_0(e^{j\omega T}) = \frac{e^{-j(N_A\omega T + 2\Phi_C(N\omega T))}}{|C(W_N^{\alpha N} e^{jN\omega T})|^2} \sum_{k=0}^{N-1} |A_k(e^{j\omega T})|^2 \tag{3}$$

[2] N is chosen to be even to allow an efficient implementation with a half-band IIR filter; see Section 4.1.

[3] We note that with only one subband user, no reallocation can be done.

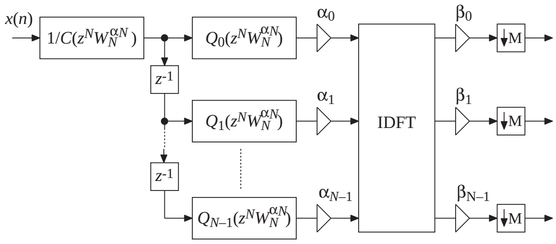

Figure 2: General realization of the analysis FB.

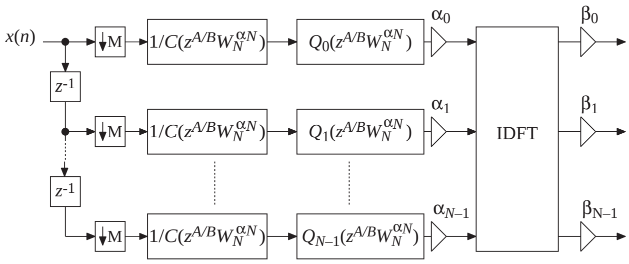

Figure 3: Realization of the analysis FB when the filters work at the lowest sampling rate available.

where $\Phi_C(\omega T)$ is the phase response of $C(W_N^{\alpha N} e^{j\omega T})$. From (3) it readily follows that the magnitude and phase responses of $V_0(z)$ are given by

$$|V_0(e^{j\omega T})| = \frac{1}{|C(W_N^{\alpha N} e^{jN\omega T})|^2} \sum_{k=0}^{N-1} |A_k(e^{j\omega T})|^2 \tag{4}$$

$$\Phi_V(\omega T) = -N_A\omega T - 2\Phi_C(N\omega T). \tag{5}$$

We note that the non-linear phase of (5) is due only to $\Phi_C(N\omega T)$.

Since the FB is oversampled, alias emanating from the decimation process can be suppressed. The aliasing terms are bounded by the stopband attenuation of the analysis filters and can be made arbitrary small by properly designing the prototype filter.

3. REALIZATION

In Fig. 2, a general realization of the analysis FB is shown. The filters $Q_p(z)$, $p = 0, 1, \ldots, N-1$ are the polyphase components of $A(z)$ according to

$$A(z) = \sum_{p=0}^{N-1} z^{-p} Q_p(z^N)$$

where $\alpha_i = W_N^{-\alpha i}$. For $\alpha = 0$ and $\alpha = 0.5$, we get $W_N^{\alpha N} = 1$ and $W_N^{\alpha N} = -1$, respectively. Either of these choices will thus lead to real-valued polyphase components $Q_p(z)$ and a decreased implementation cost.

Further, since the polyphase components are functions of z^N and N is a multiple of M $(N = (A/B)M)$ it is possible to rearrange the decimators and the filters so that they work at the lowest sampling rate present with kept real coefficients. This is shown in Fig. 3. By moving the filter $1/C$ into each branch, each filter work at a lower sampling rate, which can be an advantage in terms of speed in an implementation. However, in terms of mults/sample, nothing is gained.

The synthesis FB is realized in a corresponding way using the notation $\gamma_k = \beta_k W_N^{-k}$. In Fig. 4, a straightforward implementation is shown, using the fixed FB, and in Fig. 5

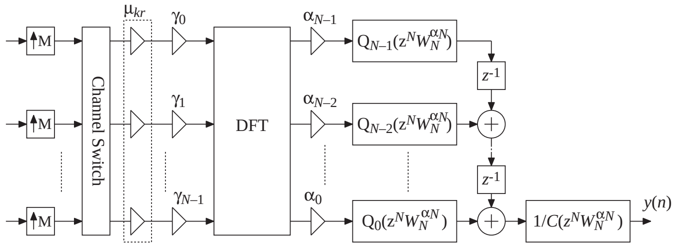

Figure 4: General realization of the synthesis FB.

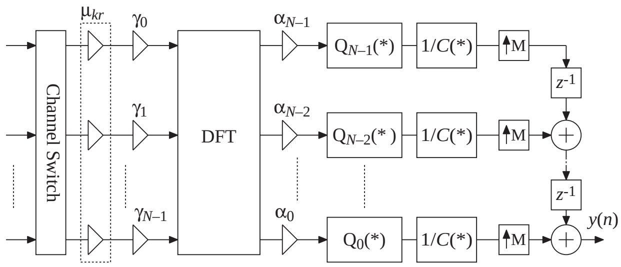

Figure 5: Realization of the synthesis FB when the filters work at the lowest sampling rate possible. In the figure, $(*)$ stands for $(z^{A/B}W_N^{\alpha N})$.

it is shown how the filters are placed so that they work at the lowest available sampling rate. Given that the delay is not critical, and thus that all μ_{kr} can be set to one, the only overhead cost compared to a fixed modulated FB is due to the channel switch[4].

4. FILTER BANK DESIGN

This section introduces a technique to design the proposed IIR FBs. We consider both NPMR and NPR. By choosing an NPMR approach, the arithmetic complexity of the on-board processing can be minimized, and if necessary, a phase equalization can be made later at the receiver on earth. For flexible FBR networks, the number of conditions to satisfy is substantially larger than for regular FBs and it is not possible in practice to design PR FBs fulfilling all these conditions. For example, there are specifications for all possible numbers of users, subbands, combinations and reallocations schemes (for details, see [2]). It has been shown in [2], that by solving the minimax problem below, there will be a slight over-design, but on the other hand, the optimization becomes very straightforward. In the section on NPMR design below, it is shown how to find a good initial solution to this nonlinear problem and Section 4.2 explains how to design the optional phase equalizer.

4.1 NPMR design

Following the design pocedure in [2] the prototype filter is designed to satisfy

$$||V_0(e^{j\omega T})|-1| \leq \delta_0, \qquad \omega T \in [0,\pi] \tag{6}$$

$$|P(e^{j\omega T})| \leq \frac{\delta_1}{N}, \qquad \omega T \in [\frac{\pi}{N}+\Delta,\pi] \tag{7}$$

where Δ denotes half the transition bandwidth of the prototype filter $P(z)$. The parameters δ_0 and δ_1 are prescribed distortion and aliasing errors. The number of conditions are thus reduced to only two, and these can be achieved by solving the following minimax optimization problem:

$$\begin{aligned} &\text{minimize} \quad \delta \\ &\text{subject to} \quad ||V_0(e^{j\omega T})|-1| \leq \delta, \quad \omega T \in [0,\pi] \\ &\qquad |P(e^{j\omega T})| \leq \delta(\tfrac{\delta_1}{N\delta_0}), \quad \omega T \in [\frac{\pi}{N}+\Delta,\pi]. \end{aligned} \tag{8}$$

The specifications in (6) and (7) are met when $\delta \leq \delta_0$. Since (8) is here a nonlinear optimization, it is crucial to have a good solution to start with. Utilizing (4), and ignoring higher-order terms such as squared stopband ripples, it can be shown that (6) and (7) are satisfied if the prototype filter $P(z)$ fulfills

$$1-0.5\delta_0 \leq |P(e^{j\omega T})| \leq 1+0.5\delta_0, \quad \omega T \in [0,\frac{\pi}{N}-\Delta]$$

$$|P(e^{j\omega T})| \leq \delta_1/N, \quad \omega T \in [\frac{\pi}{N}+\Delta,\pi]$$

$$1-\delta_0 \leq |P|^2+|P_{sh}|^2 \leq 1+\delta_0, \quad \omega T \in [\frac{\pi}{N}-\Delta,\frac{\pi}{N}]$$

where $P=P(e^{j\omega T})$ and $P_{sh}=P(e^{j\omega T}W_N)$. A convenient way to design $P(z)$ to meet the specification above is to make use of narrow-band frequency-masking techniques for IIR filters [7], [8]. This allows $P(z)$ to be designed with a sharp transition band and few distinct coefficients. We therefore express $P(z)$ as

$$P(z)=E(z^{N/2})S(z)=\frac{F(z^{N/2})S(z)}{C(z^N)} \tag{9}$$

whereby $A(z)$ in (1) becomes $A(z)=F(z^{N/2})S(z)$. By choosing $P(z)$ as in (9), it is obvious that N must be restricted to even integers. The functions $F(z)$ and $zC(z^2)$ are the numerator and denominator polynomials, respectively, of an N_Eth-order half-band IIR filter, $E(z)$, [4], where N_E is odd. (In other words, $C(z^2)$ is the denominator polynomial less the pole at the origin.) The polynomial $F(z)$ is thus an odd-order linear-phase FIR filter whereas $C(z^2)$ is an even-order filter. Further, $S(z)$ is an N_Sth-order linear-phase FIR filter. Thereby, $A(z)$ will be a linear-phase FIR filter of order $N_A=\frac{N}{2}N_E+N_S$. The functions $E(z)$ and $S(z)$ are referred to as model and masking filters respectively. The role of $S(z)$ is to remove the $N/2-1$ images present due to the factor $z^{N/2}$ in the polynomial $E(z^{N/2})$ in (9) [8]. In this way, the prototype filter can be made narrow-band and at the same time be implemented with a low arithmetic complexity, as will be illustrated in Section 5. It can be shown that a simple way to meet the specification of the prototype filter is to design $E(z)$ and $S(z)$ to fulfill

$$|E(e^{j\omega T})| \leq \delta_1/N, \quad \omega T \in [\frac{\pi}{2}+\frac{N}{2}\Delta,\pi]$$

with $E(1)=1$, and

$$1-0.5\delta_0 \leq |S(e^{j\omega T})| \leq 1+0.5\delta_0, \quad \omega T \in [0,\frac{\pi}{N}+\Delta]$$

$$|S(e^{j\omega T})| \leq \delta_1/N, \quad \omega T \in [\frac{3\pi}{N}-\Delta,\pi].$$

The filters can be designed with conventional methods [9], [10] and combined to form an initial solution to the nonlinear

[4] In this efficient realization, the separate $y_r(n)$ are not available. This is however not a problem, since only the composite output $y(n)$ is supposed to be transmitted.

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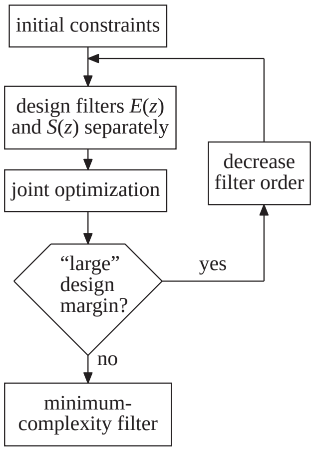

Figure 6: Flow chart over the NPMR design procedure.

minimax problem (8). The minimax optimization is thus a joint optimization of the subfilters $E(z)$ and $S(z)$.

After the joint minimax optimization, it might be possible to reduce the filter orders of the subfilters $E(z)$ and $S(z)$ and still fulfill (6) and (7). This must be done in order to ensure that the minimum-complexity prototype filter fulfilling the specification is found. A flow chart of the NPMR design procedure is shown in Fig. 6.

4.2 NPR design

In the design procedure above, the phase response of the distortion function was ignored which means that it may be too nonlinear for the application at hand. One simple way to improve the phase linearity is to equalize it with the aid of an allpass filter. Since such a filter has a constant magnitude response for all frequencies it will not affect the magnitude response of the distortion and aliasing functions. It thus suffices to consider only the phase response in this step.

Here, it follows from (5) that the nonlinearity in the phase response $\Phi_V(\omega T)$ emanates from $\Phi_C(N\omega T)$ which is the phase response of the denominator polynomial $C(z^N W_N^{\alpha N})$. The problem thus reduces to that of equalizing $\Phi_C(N\omega T)$. Further, since this is a function with a periodicity of $2\pi/N$, we use an allpass filter with transfer function $H_{AP}(z^N)$ as equalizer. The allpass filter is designed so as to minimize the maximum magnitude of the phase error $\Phi_e(\omega T)$ which is given by

$$\Phi_e(\omega T) = \Phi_V(\omega T) + \Phi_{AP}(N\omega T) + K\omega T$$

where $\Phi_{AP}(\omega T)$ is the phase response of $H_{AP}(z)$. The overall filter bank delay after the allpass equalization is $K = N_A + N \times N_{AP}$ where N_{AP} is the order of $H_{AP}(z)$. It is well known that the equiripple solution minimizes $|\Phi_e(\omega T)|$ [11]. There are many different techniques available to find this solution but we use the algorithm in [12].

5. DESIGN EXAMPLES

To demonstrate the proposed design method, two examples are given. In the first one δ_0 and δ_1 are equal to 0.01, and in the second one $\delta_0 = \delta_1 = 0.001$. The rest of the FB specification is the same as for the example in [2], which means that $M = 4$, $N = 8$, $Q = 4$, $\Delta = 0.125\pi/Q$, and $\alpha = 0.5$. The simplest channel switch $[G_k(z) = H_k(z)]$ is used since the choice of switching does not affect the orders of the errors in the FB. Each of the examples considers both the NPR and the NPMR designs and are compared with the FIR case (that is with $C(z) = 1$). Implementation complexity (the number of multiplications per input/output sample, in short mults/sample), design complexity (the number of distinct coefficients to optimize), and the total delay for the different FBs are studied and compared [5]. For the NPMR design the measure of implementation complexity is given by

$$2\left(\frac{MN_E + N_S + 1}{M} + \frac{N_E - 1}{2}\right) = 2\left(\frac{N_A + 1}{M} + N_C\right) \quad (10)$$

mults/sample, and the one for design complexity is

$$\left\lfloor \frac{MN_E + N_S}{2} + 1 \right\rfloor + \frac{N_E - 1}{2} = \left\lfloor \frac{N_A}{2} + 1 \right\rfloor + N_C \quad (11)$$

distinct coefficients. Further, the total delay through the FB is

$$2(N_A/2) = N_A \quad (12)$$

samples. For the NPR design, N_{AP} must be added to (10) and (11), whereas $N \times N_{AP}$ must be added to (12). Note also that the regular FIR design is a special case of the NPMR measures with $N_C = 0$.

	N_A	N_C	mults/sample	# of coeffs	delay
NPMR	42	2	25.5	24	42
NPMR	34	3	23.5	21	34
NPMR	39	4	28	24	39
NPR	42	2	45.5	44	202
NPR	34	3	44.5	42	202
FIR	119	–	60	60	119

Table 1: Comparison with $\delta_0 = \delta_1 = 0.01$, Example 1. For the NPR FBs, the phase error is designed to be less than 0.01. Since the NPMR FB with $N_C = 4$ did not give a good solution, it was not studied with phase equalizer.

5.1 Example 1

The best NPMR design fulfilling the specification in all aspects studied is the one with $N_C = 3$ and $N_A = 34$. In Fig. 7 the magnitude responses of the analysis (and synthesis) filters are shown. Their passband ripples are less than 0.005 and stopband ripple less than 0.0012. Figure 8 shows the distortion function which has an error less than 0.0093 [6]. The aliasing functions are shown in Fig. 9 and have a maximum value of -51.4 dB. An NPR design can be achieved by appending an allpass filter to the FB. In Fig. 10, the phase error for $N_C = 3$ and $N_A = 34$ is plotted as a function of N_{AP}. For example, if a phase error less than 0.01 is required, an allpass filter of order 21 is needed. As a comparison, a corresponding FIR filter (with $N_C = 0$) would need a filter order of 119 to fulfill the same specification. Figures of interest are summarized in Table 1.

5.2 Example 2

In this example, $\delta_0 = \delta_1 = 0.001$ and the NPMR FB with the lowest arithmetic as well as design complexity was found for $N_C = 4$ and $N_A = 50$. Figures of interest are shown in Table 2, where two NPR specifications are shown. One with phase error less than 0.01 and one with 0.001. As seen in Tables 1 and 2, the proposed IIR FBs require fewer mults/sample and distinct coefficients than the FIR FBs,

[5] One should keep in mind that the FIR FBs have the additional feature of providing an overall linear phase response.

[6] The large variations in Fig. 8 is caused by the fact that $\delta_0 = \delta_1$, and thus that the optimizations of $P(e^{j\omega T})$ and $V_0(e^{j\omega T})$ are weighted differently [see (6) and (7)].

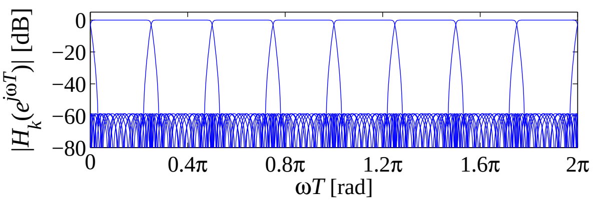

Figure 7: The magnitude responses of the analysis filters in Example 1.

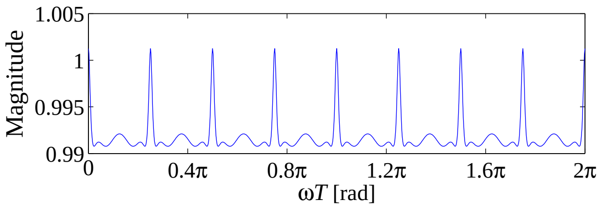

Figure 8: Magnitude of the distortion function in Example 1. See footnote 6.

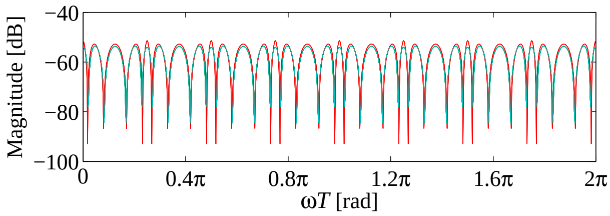

Figure 9: Magnitude of the three aliasing functions in Example 1.

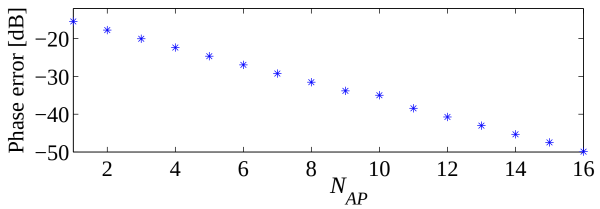

Figure 10: Maximum phase error for $N_C = 3$ and $N_A = 34$ as a function of the allpass filter order N_{AP} in Example 1.

	N_A	N_C	N_{AP}	m/s	coeffs	delay
NPMR	68	3	–	40.5	38	68
NPMR	50	4	–	33.5	30	50
NPMR	50	5	–	35.5	31	50
NPR (0.01)	68	3	26	66.5	64	276
NPR (0.01)	50	4	28	61.5	58	274
NPR (0.01)	50	5	33	68.5	64	314
NPR (0.001)	68	3	38	78.5	76	354
NPR (0.001)	50	4	41	74.5	71	378
NPR (0.001)	50	5	46	81.5	77	418
FIR	171	–	–	86	86	171

Table 2: Comparison with $\delta_0 = \delta_1 = 0.001$, Example 2. On the top row, 'm/s' stands for number of mults/sample and 'coeffs' denotes the number of distinct coefficients to optimize. The NPR FBs are designed with phase errors of both 0.01 and 0.001.

even with the smaller phase error. However, the proposed method becomes of course more interesting if larger phase errors can be allowed.

6. CONCLUSIONS

This paper introduced a class of oversampled complex-modulated causal IIR FBs for flexible frequency-band reallocation networks. In the simplest case, they have near perfect magnitude reconstruction (NPMR), but by adding a phase equalizer they can achieve near-PR. A general design procedure for the FBs was given and design examples showed that the implementation and design complexities are reduced compared to the corresponding FIR solution.

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