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ON DETECTION OF A REAL SINUSOID WITH SHORT DATA RECORD

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ABSTRACT

In this paper, we investigate two binary detection problems for a single real tone in additive white Gaussian noise using short data records. In the first hypothesis-testing scenario, we decide if a sinusoid is present in the received signal where both cases of known and unknown sinusoidal frequency will be examined. In the second problem, differentiation between two distinct noisy tones is considered. Simulation results show that the nonlinear least squares and maximum likelihood methods give identical detection performance and they outperform the periodogram approach.

1. INTRODUCTION

Detection of sinusoidal signals is of interest in many fields [1]-[3] such as radar, sonar, communications and spectroscopy. In this paper, we study two fundamental binary detection problems for a single real tone in additive white Gaussian noise using short data records. The first problem has applications in radar and sonar where we need to detect whether a sinusoid is present in the received signal or not, and it is formulated as follows. Given N samples of a received signal $x[n]$, $n = 0,1,...,N-1$, where N is small, it is required to decide between the hypotheses:

$$\begin{aligned} H_0: &\quad x[n] = q[n] \\ H_1: &\quad x[n] = A_1\cos(\omega_1 n + \phi_1) + q[n] \end{aligned} \tag{1}$$

where $q[n]$ is a white Gaussian process while A_1, ω_1 and ϕ_1 represent the tone amplitude, frequency and phase, respectively. We consider unknown A_1 and ϕ_1 while ω_1 is either known or unknown. The first hypothesis H_0 assumes that $x[n]$ consists only of noise while in H_1, the sinusoid is presumed to be present.

The second detection problem is to classify the sinusoid from two choices and this corresponds to binary communication application, although extension to multiple signals is straightforward. That is, given the received sequence $x[n]$, a decision has to be made between the hypotheses:

$$\begin{aligned} H_0: &\quad x[n] = A_0\cos(\omega_0 n + \phi_0) + q[n] \\ H_1: &\quad x[n] = A_1\cos(\omega_1 n + \phi_1) + q[n] \end{aligned} \tag{2}$$

where A_0, ω_0, ϕ_0 are the tone amplitude, frequency and phase of another sinusoid with $\omega_0 \neq \omega_1$. Here, we consider unknown amplitudes and phases but known frequencies.

In this work, we study the performance of three frequency estimation approaches, namely, periodogram, maximum likelihood (ML) estimator and nonlinear least squares (NLS) method on the binary sinusoid detection problems. It is well known that [4] the periodogram peak corresponds to the ML estimate of frequency if $x[n]$ is a noisy complex sinusoid. However, its optimality is only approximately true for $N >> 1$ in the case of a real-valued sinusoid. It is because the real tone is a sum of two complex exponentials and as a result the frequency estimate provided by the periodogram is generally biased due to interference from the negative spectral line. Furthermore, when the frequency is approaching 0 or π, the separation between the positive and negative spectral lines becomes smaller and the periodogram will not be able to resolve them if the data length is small enough. In fact, an exact ML estimator for a single real tone has been derived by Kenefic and Nutall [5]. On the other hand, in the presence of white Gaussian $q[n]$, the NLS method [6] can also be interpreted as the ML estimator, although the realizations are different.

The rest of the paper is organized as follows. In Section 2, the periodogram, ML and NLS methods are reviewed for frequency estimation. They are then utilized for the two sinusoid detection problems and the algorithms are given in Section 3. Numerical examples are presented in Section 4 to contrast the detection performance of the three approaches. Finally, concluding remarks are included in Section 5.

2. FREQUENCY ESTIMATORS

Assuming that $x[n] = A_0\cos(\omega_0 n + \phi_0) + q[n]$ where all A_0, ω_0 and ϕ_0 are unknown constants, the frequency estimate based on the periodogram, denoted by $\hat{\omega}_P$, is

$$\hat{\omega}_P = \arg\max_{\omega}\{P_x(\omega)\} \tag{3}$$

where

$$P_x(\omega) = \frac{1}{N}|X(\omega)|^2, \quad X(\omega) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \tag{4}$$

The $P_x(\omega)$ is called the periodogram of $x[n]$ while $X(\omega)$ denotes the discrete-time Fourier transform (DTFT) of $x[n]$. Since the periodogram is a nonlinear function in ω and contains multiple maxima, we can get a coarse frequency from the discrete Fourier transform (DFT) peak first and then perform the peak search to reduce computations [7]. Decomposing $X(\omega)$ into signal and noise components yields

$$X(\omega) = S(\omega) + \sum_{n=0}^{N-1} q[n]e^{-j\omega n} \tag{5}$$

where $S(\omega)$ represents the DTFT of the sinusoid and it is calculated as

$$\begin{aligned} S(\omega) &= \sum_{n=0}^{N-1} A_0 \cos(\omega_0 n + \phi_0) e^{-j\omega n} \\ &= \frac{A_0}{2} e^{j\phi_0} \sum_{n=0}^{N-1} e^{j(\omega_0-\omega)n} + \frac{A_0}{2} e^{-j\phi_0} \sum_{n=0}^{N-1} e^{-j(\omega_0+\omega)n} \\ &= \frac{A_0}{2} e^{j\phi_0} \frac{1-e^{j(\omega_0-\omega)N}}{1-e^{j(\omega_0-\omega)}} + \frac{A_0}{2} e^{-j\phi_0} \frac{1-e^{-j(\omega_0+\omega)N}}{1-e^{-j(\omega_0+\omega)}} \\ &= \frac{A_0}{2} e^{j(\phi_0-(\omega_0-\omega)(N-1)/2)} \frac{\sin(\frac{(\omega_0-\omega)N}{2})}{\sin(\frac{\omega_0-\omega}{2})} + \\ & \quad \frac{A_0}{2} e^{-j(\phi_0+(\omega_0+\omega)(N-1)/2)} \frac{\sin(\frac{(\omega_0+\omega)N}{2})}{\sin(\frac{\omega_0+\omega}{2})} \end{aligned} \quad (6)$$

Substituting (4)-(6) into (3) and then taking the expected value, the mean value of $\hat{\omega}_\text{P}$, $E\{\hat{\omega}_\text{P}\}$, is obtained as

$$E\{\hat{\omega}_\text{P}\} = \arg\max_{\omega} \left\{ |S(\omega)|^2 + N\sigma_q^2 \right\} \quad (7)$$

where σ_q^2 represents the variance of $q[n]$. It is clear from (6) and (7) that $E\{\hat{\omega}_\text{P}\}$ does not depend on the tone amplitude and noise power but is a nonlinear function of N, ω_0 and ϕ. Due to interference from the negative frequency component in $S(\omega)$, $E\{\hat{\omega}_\text{P}\}$ is generally a biased estimate of ω_0, although the first term of (6) does peak at $\omega = \omega_0$.

In fact, based on maximizing the conditional probability density of $x[n]$, $n = 0, 1, \cdots, N-1$, given A_0, ω_0 and ϕ_0, the ML estimate of ω_0 for arbitrary data record lengths in white Gaussian noise has been derived in [5]. The ML frequency estimate, denoted by $\hat{\omega}_\text{ML}$, is the value of ω which maximizes $C_\text{ML}(\omega)$:

$$\begin{aligned} & C_\text{ML}(\omega) \\ &= \frac{a_{22}(\omega)I^2(\omega) - 2a_{12}(\omega)I(\omega)Q(\omega) + a_{11}(\omega)Q^2(\omega)}{a_{11}(\omega)a_{22}(\omega) - a_{12}^2(\omega)} \end{aligned} \quad (8)$$

where $I(\omega) = \sum_{n=0}^{N-1} x[n]\cos(n\omega)$, $Q(\omega) = \sum_{n=0}^{N-1} x[n]\sin(n\omega)$, $a_{11}(\omega) = \sum_{n=0}^{N-1} \cos^2(n\omega)$, $a_{12}(\omega) = \sum_{n=0}^{N-1} \sin(n\omega)\cos(n\omega)$ and $a_{22}(\omega) = \sum_{n=0}^{N-1} \sin^2(n\omega)$. Since the cost function in (8) is highly nonlinear and multimodal, extensive computations will be involved in the optimal estimation.

On the other hand, an intuitively appealing approach to frequency estimation, based on the nonlinear regression model [6], consists of finding the unknown parameters via minimizing the NLS cost function:

$$\sum_{n=0}^{N-1} (x[n] - A_0 \cos(\omega_0 + \phi_0))^2 \quad (9)$$

The nuisance parameters of amplitude and phase can be removed and the NLS frequency estimate, denoted by $\hat{\omega}_\text{NLS}$, is the value of ω which maximizes $C_\text{NLS}(\omega)$ [8]:

$$C_\text{NLS}(\omega) = \mathbf{x}^T \mathbf{\Phi}^T(\omega) \mathbf{R}^{-1}(\omega) \mathbf{\Phi}(\omega) \mathbf{x} \quad (10)$$

where

$$\mathbf{x} = [x[0]\ x[1] \cdots x[n-1]]^T$$

$$\mathbf{\Phi} = \begin{bmatrix} \sin(0) & \sin(\omega) & \cdots & \sin((N-1)\omega) \\ \cos(0) & \cos(\omega) & \cdots & \cos((N-1)\omega) \end{bmatrix}$$

$$\mathbf{R}(\omega) = \begin{bmatrix} \sum_{n=0}^{N-1} \sin^2(\omega n) & \sum_{n=0}^{N-1} \sin(\omega n)\cos(\omega n) \\ \sum_{n=0}^{N-1} \sin(\omega n)\cos(\omega n) & \sum_{n=0}^{N-1} \cos^2(\omega n) \end{bmatrix}$$

As in (8), the cost function of (10) is highly nonlinear and thus extensive computations will be required for the peak search. Nevertheless, by expanding $C_{NLS}(\omega)$, we easily get $C_{NLS}(\omega) = C_{ML}(\omega)$ which implies that $\hat{\omega}_{NLS} = \hat{\omega}_{ML}$ and thus the NLS approach also achieves optimum estimation performance.

3. DETECTION ALGORITHMS

In the first hypothesis-testing scenario of (1), we decide whether the received signal consists of noise only or is a noisy sinusoid. When the frequency ω_1 is known, we compute $P_x(\omega_1)$, $C_\text{ML}(\omega_1)$ and $C_\text{NLS}(\omega_1)$ and then compare these values with their corresponding thresholds. If the peak coefficient is larger than the threshold, H_1 is accepted, otherwise H_0 is chosen. Assuming that the data independent elements have been precomputed, the computational requirements for the periodogram, ML and NLS estimators are $2N+2$ multiplications and $2N-1$ additions, $2N+7$ multiplications and $2N$ additions, and N^2+N multiplications and N^2 additions, respectively. Although both ML and NLS methods give optimum performance, we see that the former is much more computationally attractive, particularly for larger N. When the frequency is unknown, we use $P_x(\hat{\omega}_\text{P})$, $C_\text{ML}(\hat{\omega}_\text{ML})$ and $C_\text{NLS}(\hat{\omega}_\text{NLS})$ instead and thus complex peak search procedure is required. In our study, we first use the DFT peak as the coarse frequency estimate in all methods and then apply golden section search for fine estimation.

In the second hypothesis-testing scenario of (2), we need to determine whether the frequency of the received sinusoid is of ω_0 or ω_1. When the frequencies are available, we compute $P_x(\omega_0)$ and $P_x(\omega_1)$ for the periodogram approach. If $P_x(\omega_1) > P_x(\omega_0)$, we choose H_1, otherwise, H_0 is accepted. In a similar manner, $C_\text{ML}(\omega_0)$, $C_\text{ML}(\omega_1)$, $C_\text{NLS}(\omega_0)$ and $C_\text{NLS}(\omega_1)$ are computed in the ML and NLS based detectors.

4. NUMERICAL EXAMPLES

Computer experiments have been conducted to compare the detection performance of the periodogram, ML and NLS methods for the two hypothesis-testings in (1) and (2). The tone amplitudes A_0 and A_1 are set to be identical, the phases ϕ_0 and ϕ_1 are uniformly distributed between 0 and 2π at each trial, and $\omega_0 = 11\pi/12$ and $\omega_1 = 6\pi/12$. The signal-to-noise ratio (SNR) is defined as $A_0^2/(2\sigma_q^2)$ and $N = 4$ is considered. All results are averages of 50000 independent runs.

Figures 1 to 3 show the receiver operating characteristic (ROC), that is, probability of detection versus false alarm rate, in detecting a pure sinusoid in the presence of white Gaussian noise based on the periodogram, ML and NLS methods, respectively. The simulation results of the operating characteristics are obtained by using the method suggested in [9]. Four different SNR values, namely, -5 dB, 0 dB, 5 dB and 10 dB are considered and ω_1 is assumed known. As expected, the ML and NLS methods provide identical results. Interestingly, the periodogram gives the same detection performance as well. The above test is repeated for unknown ω_1 and the results are plotted in Figures 4 to 6. It is observed that ML and NLS methods give the same detection performance again and outperform the periodogram. A possible reason for the inferiority of the periodogram is that it provides biased real-tone frequency estimation particularly for short data lengths.

Figures 7 and 8 show the detection probability of ω_0 and ω_1, respectively, versus SNR for the binary classification. This test in fact corresponds to the caller ID signal decoding problem with sinusoidal frequencies of 2200 Hz and 1200 Hz, which represent bits 0 and 1, respectively, at a sampling frequency of 4800 Hz [8]. Again, we see that both ML and NLS detectors give the same performance for both bits 0 and 1. In classifying ω_0, the periodogram cannot provide a detection probability of one even for sufficiently high SNR because the frequency is close to π such that it is unable to resolve the peaks of positive and negative spectral lines for some phase angles. It is also observed that the periodogram is biased in the sense that bit 1 is preferred over bit 0 for all SNR conditions, although it has slightly higher probability of detection than those of ML and NLS detectors for bit 1.

5. CONCLUDING REMARKS

The periodogram, maximum likelihood (ML) and nonlinear least squares (NLS) methods have been studied for deciding if a sinusoid is present as well as differentiating between two distinct noisy tones for short data records. It is shown that the ML and NLS detectors give the same performance for the two hypothesis-testing scenarios but the former should be preferred because of its smaller computational requirement. Moreover, apart from sinusoidal detection with known frequency, the periodogram is inferior to the ML and NLS methods. It is noteworthy that the results hold for larger data lengths and the multiple sinusoidal signal classification problem as well, where the ML method is always the best choice. An interesting extension of this work is to produce the theoretical ROC of the ML detector.

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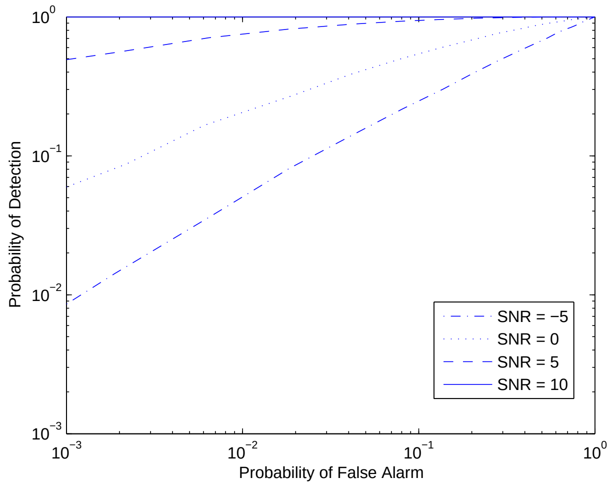

Figure 1: ROC of periodogram with known frequency

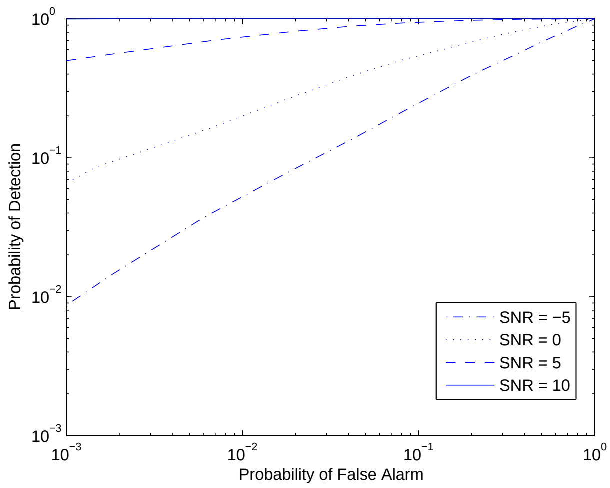

Figure 2: ROC of ML estimator with known frequency

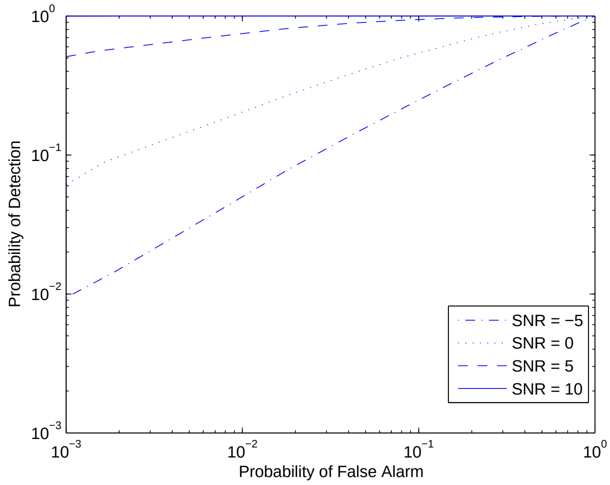

Figure 3: ROC of NLS estimator with known frequency

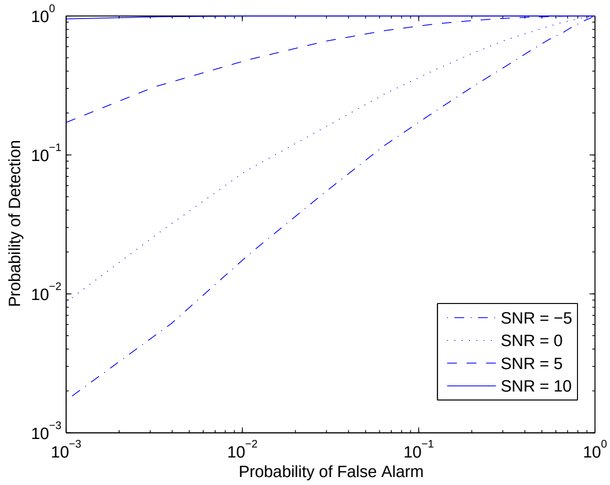

Figure 4: ROC of periodogram with unknown frequency

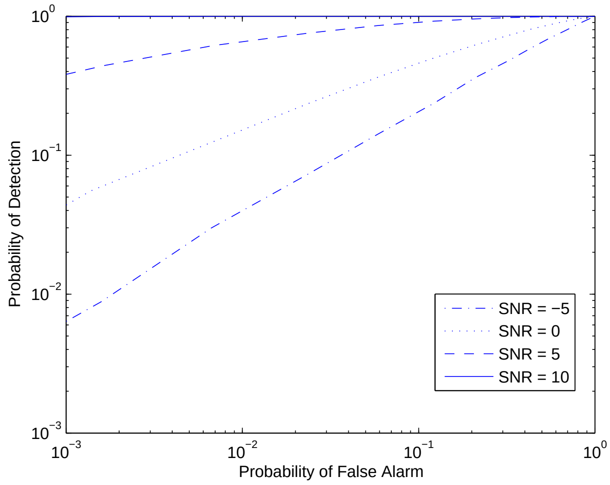

Figure 5: ROC of ML estimator with unknown frequency

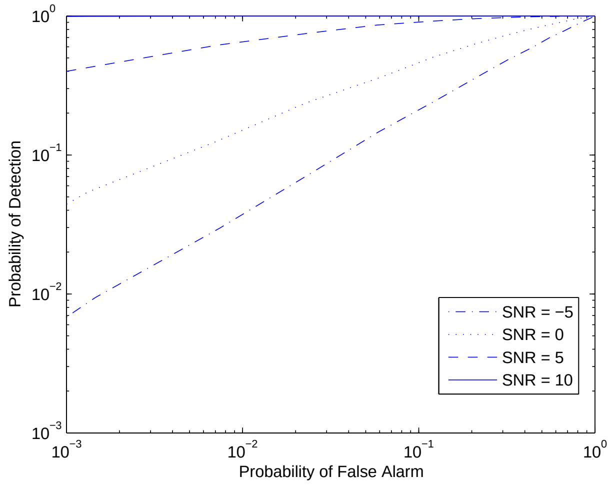

Figure 6: ROC of NLS estimator with unknown frequency

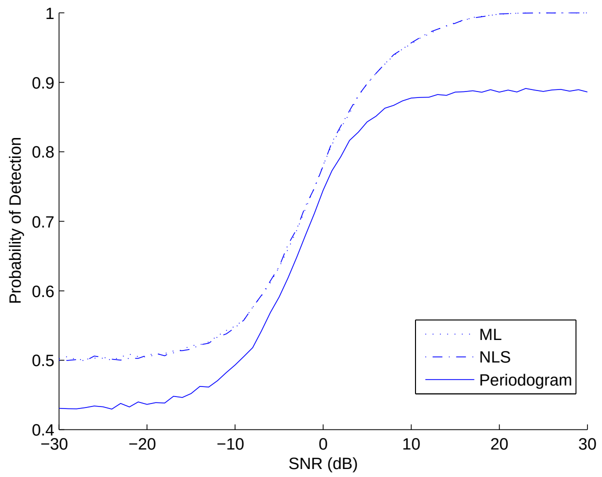

Figure 7: Probability of detection for bit 0

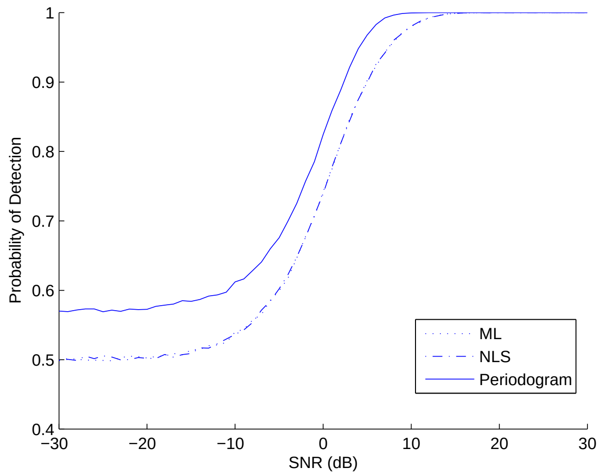

Figure 8: Probability of detection for bit 1