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ON THE TIME INVARIANCE OF LINEAR SYSTEMS

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ABSTRACT

In this paper we revise the concept of time invariance of a system, using exclusively a time domain approach. To this aim, we adopt a group theoretical formulation which let us recognize that the concept of time invariance is truly confined to a few possible cases. This depends on the fact that, for a general multirate linear system, the shift invariance property depends not only on the kernel, but also on the input and output domains. We illustrate the concept with an example of application in onedimensional domains, indicating that our definition has a useful impact in the analysis and synthesis of multirate linear systems. Furthermore, the proposed approach permits to extend the concept of system invariance to multidimensional domains.

1. INTRODUCTION

In this paper we present a unified framework for the representation of time invariance in linear systems (LSs). Several papers in the past addressed the problem of time invariance, most of them dealing with periodical invariance, also called periodical variance. For standard references on linear time varying systems, see [1] [2] [3].

Usually, the definition of a periodical system focuses on the case of equal input and output domains, i.e. single rate systems [1] [2] [4]. In this case, in fact, the periodical invariance of a system has a counterpart into the periodicity of the system impulse response [1] [2] or of the kernel [3] [4]. According to [4], the definition of periodical invariance is possible even in the case of multidimensional discrete linear systems with equal outer domains. The early work [3] introduced a complete state model of periodically time varying systems, deriving also the spectral representation of multirate filters where the input sampling rate is p times larger than the output sampling rate. But, in the most general case of different outer domains, i.e. multirate systems, the definition of shift invariance depends on both the input and the output domain.

Recently, the authors of [5][6] related the periodical invariance of a multirate system to the ratio of the input and output sample rates. They introduced the concept of (m,n)-shift invariance, exploiting the reasoning that if the input sequence is delayed by n samples, then the output sequence will be delayed by m samples.

The main contribution of this work is to provide a unified definition of shift invariance which works in the most general case. In particular, we do not assume any *a priori* decomposition of the system in order to introduce the concept of periodicity and invariance. An appealing feature of such a definition is that the concept of invariance for a multirate system can be extended naturally to multidimensional systems [7]. Also, the periodical invariance can be expressed exclusively in the time domain, through kernel's properties.

In particular we use a group theoretic approach [7][8], where the signal domain is always a subgroup of $\mathbb{R}^m$: this is the key assumption leading to a unified definition of shift invariance in terms of input/output domains only and to a *representation-independent* approach. For example, the proposed development applies also to the continuous domain, and is not limited to discrete groups [3][5]. Thus, this notion of shift invariance adapts to a larger class of systems.

2. BASIC DEFINITIONS

In what follows, the fundamental assumption we need is that the signal domain $\mathbb{L}$ is an Abelian group. Hence, a signal on $\mathbb{L}$, denoted by $s(t)$, $t \in \mathbb{L}$, is a complex function defined on the group $\mathbb{L}$.

For every $s(t)$, $t \in \mathbb{L}$, it is possible to introduce the linear operator

$$H_s = \int_{\mathbb{L}} dt\, s(t)\ , \tag{1}$$

with the shift invariance property

$$\int_{\mathbb{L}} dt\, s(t) = \int_{\mathbb{L}} dt\, s(t-t_0)\,, \qquad \forall t_0 \in \mathbb{L}\ , \tag{2}$$

which happens to be a general property of the Haar integral.

The strict mathematical formulation [9] requires that the integral appearing in (1) is the Haar integral and the signal domain a locally compact Abelian group. However, for the most cases encountered in practice, such a general topological formulation is far beyond a fairly general description and can be specialized to the cases of interest. They are represented by the real domain $\mathbb{R}$, the discrete domain $Z(T) = \{nT \mid n \in \mathbb{Z}\}$, and their multidimensional counterparts, that is $\mathbb{R}^m$ and lattices $\mathbb{L}$. In the following, some preliminary definitions are given, according to [4].

2.1 Lattices, cells and discrete signals

Definition 1 A *lattice* $\mathbb{L}$ in $\mathbb{R}^m$ of rank m is a set of the form $\mathbb{L} = \{\mathbf{L}\,\mathbf{n} \mid \mathbf{n} \in \mathbb{Z}^m\}$, where $\mathbf{L}$ is a non–singular $m \times m$ real matrix, called a *basis* of $\mathbb{L}$.

The basis $\mathbf{L}$ is not unique: indeed, if $\mathbf{E}$ is any matrix of integers such that $|\det \mathbf{E}| = 1$, then $\mathbf{LE}$ generates the same lattice $\mathbb{L}$. Moreover, all the bases of $\mathbb{L}$ can be written in this form, so that the quantity $d(\mathbb{L}) = |\det \mathbf{L}|$ is specific of the

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lattice $\mathbb{L}$ [4]. If $\mathbb{J}$ and $\mathbb{K}$ are lattices with $\mathbb{J}$ a *sublattice* of $\mathbb{K}$ ($\mathbb{J} \subset \mathbb{K}$), then $d(\mathbb{J})$ is an integer multiple of $d(\mathbb{K})$. The integer $(\mathbb{K} : \mathbb{J}) = d(\mathbb{J})/d(\mathbb{K})$ is called the index of $\mathbb{J}$ in $\mathbb{K}$. In the following we assume that the mD lattices belong to the class $\mathcal{L}_m$ of the sublattices of a same "superlattice" of the form $Z(T_1) \times \cdots \times Z(T_m)$, where $Z(T) = \{nT | n \in \mathbb{Z}\}$. Then, the bases can be written in the form $\mathrm{diag}(T_1, \cdots, T_m)\,\mathbf{A}$, where $\mathbf{A}$ is an integer matrix and the diagonal matrix can take into account physical dimensions (time spacing, horizontal and vertical spacings, etc.).

We shall illustrate the concept in 1D and 2D. In the 1D case, the general lattice of $\mathcal{L}_1$ has the form $Z(T)$, where $T > 0$ represents the time spacing of the signal. In the 2D case, we find convenient to express lattices of $\mathcal{L}_2$ in the form $Z_a^i(T_1, T_2)$, corresponding to the upper diagonal basis

$$\mathbf{L} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} a & i \\ 0 & 1 \end{bmatrix} \qquad i, a \in \mathbb{Z}, \quad 0 \le i < a. \qquad (3)$$

So, e.g., $Z_2^1(T_1, T_2)$ is a quincunx lattice and $Z_4^2(T_1, T_2)$ is a hexagonal lattice. These lattices are illustrated in Fig.1.

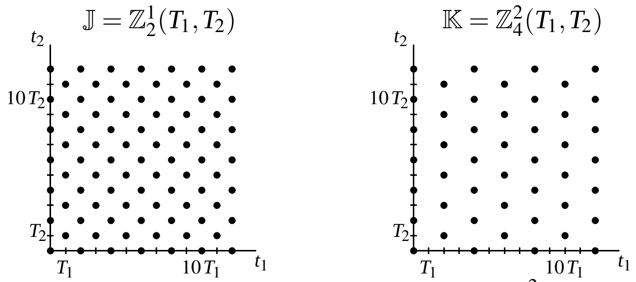

Fig. 1 $-$ Examples of 2D lattices of $\mathbb{R}^2$.

Let $\mathbb{G}$ be a lattice and let $\mathbb{J}$ be a sublattice of $\mathbb{G}$. A *coset of* $\mathbb{J}$ *in* $\mathbb{G}$ is a shifted version of $\mathbb{J}$, $\mathbb{J} + c = \{j + c \,|\, j \in \mathbb{J}\}$, $c \in \mathbb{G}$. Two cosets $\mathbb{J} + c$ and $\mathbb{J} + d$ may be distinct or coincident, and the coincidence occurs iff $c - d \in \mathbb{J}$.

Definition 2 A subset C of $\mathbb{G}$ is called a (discrete) *cell of* $\mathbb{G}$ *modulo* $\mathbb{J}$, and here denoted by $[\mathbb{G}/\mathbb{J}]$, if the cosets $\mathbb{J} + c$, $c \in C$ represent a partition of $\mathbb{G}$, that is $(\mathbb{J} + c) \cap (\mathbb{J} + d) = \emptyset$, $c \neq d$ and $\bigcup_{c \in C} (\mathbb{J} + c) = \mathbb{G}$.

Given a lattice $\mathbb{L}$, the *reciprocal* lattice is defined by $\mathbb{L}^\star = \{f \in \mathbb{R}^m | f t \in \mathbb{Z}, t \in \mathbb{L}\}$, where $f t = f_1 t_1 + \cdots + f_m t_m$ is the inner product of $f = (f_1, \cdots, f_m)$ and $t = (t_1, \cdots, t_m)$ (for simplicity we avoid the notation $f' t$). If $\mathbf{L}$ is a basis of $\mathbb{L}$, then a basis of $\mathbb{L}^\star$ is given by the inverse transpose of $\mathbf{L}$. Hence the relation $d(\mathbb{L}^\star)\, d(\mathbb{L}) = 1$. Note that for two lattices $\mathbb{J}$ and $\mathbb{K}$, if $\mathbb{J} \subset \mathbb{K}$, then $\mathbb{J}^\star \supset \mathbb{K}^\star$. For a given cell $C = [\mathbb{K}/\mathbb{J}]$ it is possible to define the *reciprocal cell* as $C^\star = [\mathbb{J}^\star/\mathbb{K}^\star)$, which has the same cardinality as C.

A mD discrete signal on $\mathbb{L}$, denoted by $s(t)$, $t \in \mathbb{L}$, is a complex function defined on the mD lattice $\mathbb{L}$. The density of the lattice given by $\mu(\mathbb{L}) = 1/d(\mathbb{L})$ will be called the *rate* of the signal. For every $s(t)$, $t \in \mathbb{L}$, the linear operator in (1) *has the same properties* as the Lebesgue integral on the real line and becomes

$$\int_{\mathbb{L}} dt\, s(t) \triangleq d(\mathbb{L}) \sum_{t \in \mathbb{L}} s(t). \qquad (4)$$

3. TIME INVARIANCE

3.1 Shift Invariance

In order to describe the concept of time invariance, we use the shift operator, applied to a signal $s(t)$, $t \in \mathbb{L}$ as $\mathcal{S}_p^{\mathbb{L}} s(t) = s(t - p)$, $p \in \mathbb{L}$. Notice that, since $\mathbb{L}$ is assumed to be an Abelian group, it results

$$\mathcal{S}_{p_1+p_2}^{\mathbb{L}} = \mathcal{S}_{p_1}^{\mathbb{L}} \mathcal{S}_{p_2}^{\mathbb{L}} = \mathcal{S}_{p_2}^{\mathbb{L}} \mathcal{S}_{p_1}^{\mathbb{L}} = \mathcal{S}_{p_2+p_1}^{\mathbb{L}}, \qquad \forall p_1, p_2 \in \mathbb{L}. \quad (5)$$

Also, the shift operator only deals with shifts belonging to the signal domain. Thus, for example, the shift $\mathcal{S}_{T/2}^{Z(T/2)}$ applies to $y(t), t \in Z(T/2)$ whereas a $T/2$ shift does not apply to $x(t), t \in Z(T)$.

Let $\mathcal{L}$ be a system having input and output signals, $x(u)$ and $y(t)$, defined on the domains $\mathbb{J}$ and $\mathbb{K}$ respectively, briefly a $\mathbb{J} \to \mathbb{K}$ system. We can state the *shift invariance* of $\mathcal{L}$ with respect to p when $\mathbb{J}$ and $\mathbb{K}$ have a common group operation, that is they must be subgroups of a same group $\mathbb{L}$ and $\mathcal{L}\, \mathcal{S}_p^{\mathbb{J}} = \mathcal{S}_p^{\mathbb{K}}\, \mathcal{L}$, $p \in \mathbb{J} \cap \mathbb{K}$. But, since the shift operator only deals with shifts belonging to the signal domain, the shift makes sense only if $p \in \mathbb{J}$ and $p \in \mathbb{K}$: we say that $p \in \mathbb{J} \cap \mathbb{K}$ is a *compatibility* condition.

Given a $\mathbb{J} \to \mathbb{K}$ system $\mathcal{L}$, we define its *subset of invariance* as the set $\mathbb{P}$ of those p such that shift invariance applies. Notice that $0 \in \mathbb{P}$ for any system $\mathcal{L}$. Also, if $\mathbb{P}$ is the subset of invariance of a LS, then $\mathbb{P}$ is always an Abelian group and, more specifically, $\mathbb{P} \subset \mathbb{J} \cap \mathbb{K}$.

The above definition holds for a general system. In the case of a linear system $\mathcal{L}_h$, the I/O relationship is specified by the kernel $h(t, u)$ through the following unified representation

$$\mathcal{L}_h : \quad y(t) = \int_{\mathbb{J}} du\, h(t, u)\, x(u), \qquad t \in \mathbb{K}, \qquad (6)$$

where the integral has been introduced before.

In the case of a linear system the set of invariance is fully specified in terms of the kernel, since from (6) it follows

Theorem 1. A $\mathbb{J} \to \mathbb{K}$ system with kernel $h(t, u)$ is shift invariant on $\mathbb{P}$ if and only if

$$h(t + p, u + p) = h(t, u), \qquad \forall\, p \in \mathbb{P}. \qquad (7)$$

The above statement represents a straightforward generalization of the analogous condition in the case of equal domains [3][4].

3.2 Classification

From the above, it follows that the general periodicity of a linear system is constrained by

$$\mathbb{O} \subseteq \mathbb{P} \subseteq \mathbb{J} \cap \mathbb{K}, \qquad (8)$$

with $\mathbb{O} \triangleq \{0\}$ representing the trivial group. Based on (8), we can introduce a corresponding classification for the *invariance* of a general linear system $\mathcal{L}$.

- $\mathbb{P} = \mathbb{O}$: time-varying system (TV)
- $\mathbb{O} \subset \mathbb{P} \subseteq \mathbb{J} \cap \mathbb{K}$: periodically invariant (PI)
- $\mathbb{P} = \mathbb{J} \cap \mathbb{K}$: quasi-invariant (QI)

- $\mathbb{P} = \mathbb{J}$: strictly invariant (SI).

We call *periodically invariant* those systems denoted *periodically variant* in literature (see for example [1][5]). This choice is dictated by the analogy with the definition of strict invariant system, which is obtained as a limit case when the periodicity coincides with the input domain. Anyway, our definition leads to a subtle distinction between the various classes of invariant systems listed above. In particular, the invariance displays a hierarchy of inclusion with respect to the periodicities. In fact, for time-varying systems it holds $\mathbb{P} = \{0\}$, whereas the strict invariance is possible only in the case $\mathbb{P} = \mathbb{J} \subset \mathbb{K}$ and, furthermore, single-rate SIL systems can be obtained if $\mathbb{P} = \mathbb{J} = \mathbb{K}$.

As for strictly invariant systems, the kernel collapses into a single argument function, as shown by the following.

Theorem 2. The kernel $h(t,u)$ of a QILS can be written in the form

$$h(t,u) = g(t-u).$$

The proof of this fact can be found in [7]. In the case of a QILS, the function $g(\cdot)$ is called the *impulse response* of the system and is defined on $\mathbb{E} = \mathbb{J} + \mathbb{K} \triangleq \{ j+k \,|\, j \in \mathbb{J}, k \in \mathbb{K}\}$.

3.3 Basic QIL and PIL systems

In the class of QILSs a further classification can be achieved, depending on domain orderings, giving

1) ordinary filters, when $\mathbb{J} = \mathbb{K}$ (single-rate systems $\mathbb{E} = \mathbb{J}$),

2) up-samplers, when $\mathbb{J} \subset \mathbb{K}$ (two-rate systems $\mathbb{E} = \mathbb{K}$),

3) down-samplers, when $\mathbb{J} \supset \mathbb{K}$ (two-rate systems $\mathbb{E} = \mathbb{J}$).

Any other QILS can be synthesized starting from these three components, which we shortly describe in the following.

An ordinary *filter* on $\mathbb{J}$ with impulse response $g(v)$, $v \in \mathbb{J}$ is a single-rate system governed by the I/O relationship

$$y(t) = \int_{\mathbb{J}} du\, g(t-u)\, x(u) = g * x(t)\,, \qquad t \in \mathbb{J}\,. \tag{9}$$

Eq. (9) represents a convolution of two signals and has the same properties as the standard convolution on the real line, e.g. the identity element is the Dirac's delta function if $\mathbb{J} = \mathbb{R}$ and, in the case of discrete domains,

$$\delta_{\mathbb{J}}(t) = \begin{cases} 1/d(\mathbb{J}) & t = 0 \\ 0 & t \neq 0 \end{cases} \quad t \in \mathbb{J}\,. \tag{10}$$

In a $\mathbb{J} \to \mathbb{K}$ *up-sampler*, since $\mathbb{J} \subset \mathbb{K}$, the impulse response, defined on $\mathbb{K}$, becomes $\delta_{\mathbb{K}}(v)$ and the I/O relationship becomes $y(t) = A_0 x(t)$ if $t \in \mathbb{J}$, $A_0 = d(\mathbb{J})/d(\mathbb{K})$ and 0 otherwise. Up-sampling retains a fixed multiple of the input signal values at every point t that $\mathbb{J}$ and $\mathbb{K}$ have in common and inserts zeros in the rest of the output domain $\mathbb{K}$.

On the contrary, in a $\mathbb{J} \to \mathbb{K}$ *down-sampler*, since $\mathbb{J} \supset \mathbb{K}$, the impulse response is $\delta_{\mathbb{J}}(u)$, $u \in \mathbb{J}$ and the following I/O relationship is achieved: $y(t) = x(t)$, $t \in \mathbb{K}$, i.e, the input signal is restricted to the output domain.

Observe that filters and up-samplers are strictly invariant systems, whereas down-samplers are QILSs.

In the broader class of PILSs, instead, other basic components can be defined, such as modulators, with $\mathbb{J} = \mathbb{K}$, which multiply the input signal by a periodic carrier. A modulator with carrier $\gamma(t)$, $t \in \mathbb{J}$ with periodicity $\mathbb{P} \subset \mathbb{J}$ has I/O relationship $y(t) = \gamma(t)x(t)$, $t \in \mathbb{J}$ and kernel given by $h(t,u) = \gamma(t)\,\delta_{\mathbb{J}}(t-u), t,u \in \mathbb{J}$ [10].

4. COMPARISON WITH PREVIOUS DEFINITIONS

In this section we compare the notion of shift invariance of literature with the definition introduced here. In the case of single rate systems, i.e. $\mathbb{J} = \mathbb{K}$, we already mentioned that shift invariance condition (7) includes also the traditional concept of *periodically varying* system, as given in [3][4]. Some authors, i.e. [1] [2], define *linear periodically time-varying* systems, with period M, through the I/O relationship

$$y(n) = \sum_m a_n(m)\, x(n-m) \tag{11}$$

where $a_n(m) = a_{n+M}(m)$, $\forall m,n$. But, it is easy to show that such a definition is equivalent to (7) for single rate systems.

Finally, according to the definition of [5][6], a multirate linear system $\mathcal{L}$ is (m,n)-shift invariant, when

$$h(k+m, l+n) = h(k,l)\,, \qquad \forall k,l \in \mathbb{Z}. \tag{12}$$

The definition in (12) is apparently more general than (7), since it relies on a different definition of shift invariance

$$\mathcal{L}\; \mathcal{S}_n^{\mathbb{Z}} = \mathcal{S}_m^{\mathbb{Z}}\; \mathcal{L}\,. \tag{13}$$

In what follows we prove that, in the case of linear systems, our definition is equivalent to (13). Preliminarily, we observe that if $\mathcal{L}$ is a (m,n)-shift invariant linear system and it is also (m',n')-shift invariant, then $m/n = m'/n'$. In fact, a (m,n)-shift invariant system is also (km,kn)-shift invariant:

$$\mathcal{L}\mathcal{S}_{kn}^{\mathbb{Z}} = \mathcal{S}_m^{\mathbb{Z}}\mathcal{L}\mathcal{S}_{(k-1)n}^{\mathbb{Z}} = \mathcal{S}_{2m}^{\mathbb{Z}}\mathcal{L}\;\mathcal{S}_{(k-2)n}^{\mathbb{Z}} = \ldots = \mathcal{S}_{km}^{\mathbb{Z}}\;\mathcal{L} \tag{14}$$

Thus, if we assume that $\mathcal{L}$ is both (m,n) and (m',n')-shift invariant, then, from (14) it is also $(n'm,nn')$ and (nm',nn') shift invariant, so that $n'm = nm'$. Also, let m_0 and n_0 are coprime and $m_0/n_0 = m/n$, then, for any other pair (m,n) such that $\mathcal{L}$ is (m,n)-shift invariant, there exist integer k such that $(m,n) = (km_0, kn_0)$. Now, let $\mathcal{L}$ have kernel $h(r,s)$, and consider the $Z(T_1) \to Z(T_2)$ multirate linear system $\tilde{\mathcal{L}}$, with kernel $h(rT_2, sT_1)$: apart from a linear transformation of the time domain, $\tilde{\mathcal{L}}$ has the same I/O relationship of the original system. If we choose T_1 and T_2 so that $m_0/n_0 = T_1/T_2$, we recognize that $\tilde{\mathcal{L}}$ is time invariant over set

$$\mathbb{P} = \{\, p \,|\, p = nT_1 = mT_2,\ \mathcal{L} \text{ is } (m,n)-\text{shift invariant}\}.$$

In general, $\mathbb{P} \subset Z(T_1) \cap Z(T_2)$: if m and n are coprime, the set $\mathbb{P} = Z(T_1) \cap Z(T_2)$ and the system is a QILS.

Formally, we proved that

Theorem 3. For every (m,n)-shift invariant multirate linear system $\mathcal{L}$ there exists a linear transformation of the time domain such that the resulting $Z(T_1) \to Z(T_2)$ multirate linear system i) has the same I/O relationship as $\mathcal{L}$ and ii) it is shift-invariant on $\mathbb{P} \subset Z(T_1) \cap Z(T_2)$.

We remark that the above result states that the overall class of (m,n) shift invariant systems can be captured by our definition through an affine transformation (rotation) according to the ratio m/n. We believe that a major advantage of the definition presented here is that it fits naturally the multidimensional domain, because it relies only on the concepts of sum and intersection of time domains, whereas (m,n)-shift invariance is inherently related to the 1-dimensional case. In the next section we show that the interest to this novel definition is not limited to the theoretical aspect of the problem, but, using our definition of shift invariance, we can obtain efficient decompositions and analysis of multirate linear systems.

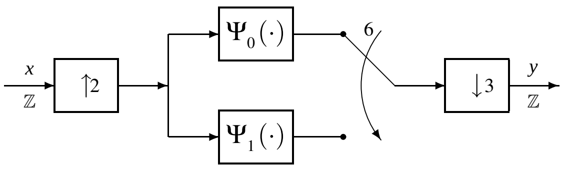

Fig. 2 – Structure of a $(4,6)$–periodic system with output switch.

5. APPLICATION TO LSTV SYSTEMS

We derive an example of decomposition of a general multirate linear system, which shows that the notion of shift invariance introduced in this paper is effective. Notice that several related works in literature [5] [6] focused on equivalent structures to represent general multirate linear systems. We provide an all time domain relationship between the components involved, which, if we would not leverage the framework presented before, would represent a much more complicated task even for the simple example reported here.

In particular, we consider a $(4,6)$-shift invariant system, which can be represented as a linear switched time varying (LSTV) system [6], as depicted in Fig.2. The switch collects 6 consecutive samples from each branch, and then connects to the next subsystem. The corresponding multirate system we examine first in this example is a $Z(2) \to Z(3)$ PIL system with kernel $h(\cdot,\cdot)$ and assigned periodicity $\mathbb{P} = Z(12) \subset Z(6) = Z(2) \cap Z(3) = \mathbb{P}_0$. Later, we will show that the decomposition can be further specialized in case the multirate system is a $Z(2) \to Z(3)$ QILS ($\mathbb{P} = \mathbb{P}_0$).

In the first step, let us decompose the output domain $Z(3)$ according to the cosets of $Z(12)$ in $Z(3)$, as $t = t_0 + a$, $t \in Z(3)$, $t_0 \in Z(12)$ and $a \in [Z(3)/Z(12)) = \{0,3,6,9\}$. Hence, the I/O relationship can be written as

$$y(t_0+a) = \int_{Z(2)} du\; h(t_0+a,u)x(u) = = \int_{Z(2)} du\; q_a(t_0+a-u)x(u), \tag{15}$$

where

$$q_a(u) = h(a, a-u)\ ,\ a \in [Z(3)/Z(12))\ ,\ u \in Z(2). \tag{16}$$

The final expression in (15) means that the components $y(t_0+a)$ of the output signal can be obtained through a bank of filters on $Z(2)$ and a bank of $Z(2) \to Z(12)$ down-samplers [2]. Finally, the output signal $y(t)$ is recovered from its components which must be up-sampled $Z(12) \to Z(3)$, properly delayed and summed as shown in Fig.3. Note that the last

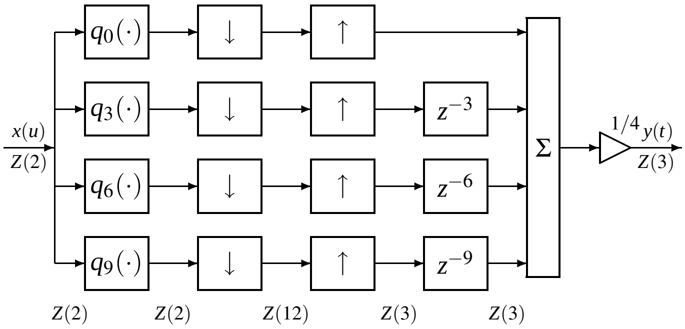

Fig. 3 – Output decomposition of a $Z(2) \to Z(3)$ PILS.

multiplier is justified to compensate the up-sampling operation $Z(12) \to Z(3)$.

For each branch of the architecture, we further refine our analysis using standard noble identities as depicted in Fig. 4 for the a-th branch. Briefly, each $Z(2) \to Z(12)$ down-sampler is split into the cascade of two down-samplers $Z(2) \to Z(6)$ and $Z(6) \to Z(12)$, and, analogously, a similar decomposition holds also for the $Z(12) \to Z(3)$ up-sampler.

Notice that this is a crucial step, since we report the decomposition to the thickest possible periodicity, i.e. $Z(6)$.

Hence, at step 3), we can replace the cascade of $Z(6) \to Z(12)$ down-sampler and $Z(6) \to Z(12)$ up-sampler with a modulator having carrier $\gamma(v)$, $v \in Z(6)$, which represents the indicating function of $Z(12)$. At step 4), the modulator and the $Z(6) \to Z(3)$ up-sampler are reversed in order, leading to an up-sampled version of the modulator carrier. Step 5) uses a noble identity to bring the system in the $Z(1)$ domain, which happens to result from the sum $Z(2)+Z(3)$. At step 5), we also move the modulator after the delay element, resulting in a modified carrier $\gamma_a(t) = 1$ if $t = a + t_0$, $t_0 \in Z(12)$, $a \in [Z(3)/Z(12))$, and zero otherwise. At step 6), we apply

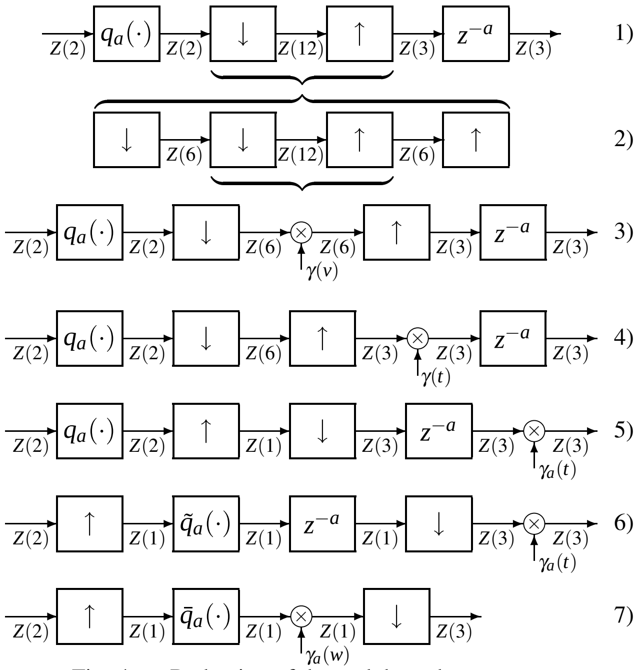

Fig. 4 – Reduction of the a–th branch.

another noble identity to reverse in order the filter $q_a(\cdot)$ and the successive up-sampler. In addition, we reverse in order the $Z(1) \to Z(3)$ down-sampler and the delay element.

In the end, we obtain the structure in 7), where the modulator function is an up-sampled version of that defined at step 5). Furthermore, each $\tilde{q}_a(w)$ is the $Z(2) \to Z(1)$ up-sampled version of $q_a(\cdot)$, whereas $\bar{q}_a(w) = \tilde{q}_a(w-a)$. Now, we can use (16) and provide the explicit form of $\bar{q}_a(w)$, which can be easily expressed in terms of the original kernel

$$\bar{q}_a(w) = \begin{cases} 2h(a, 2a-w) & \text{if } w \in Z(2) \\ 0 & \text{otherwise} \end{cases}$$

The overall final scheme is depicted in Fig.5.

At this point we can relate our final scheme to that of Fig. 2. In particular, the output switch can be replaced by a bank of modulators having modulating functions $\Gamma_0(w) =$

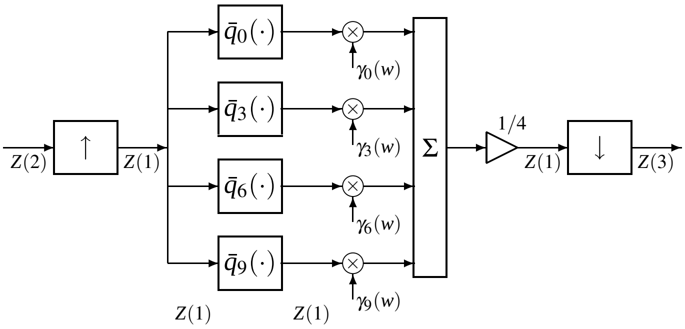

Fig. 5 – Final scheme.

$\sum_{k=0}^{5}\gamma(w-k)$, $w\in Z(1)$ and $\Gamma_6(w)=\Gamma_0(w-6)$ and a summation element. In fact, it is sufficient to note that function $\Gamma_0(w)$ annihilates the output of lower two branches of our scheme and $\Gamma_6(w)$ annihilates the output of upper two branches, respectively (Fig. 6). Thus, the scheme of Fig. 2 can be obtained from our decomposition, letting $\Psi_0(w)=\gamma_0\,\bar{q}_0(w)+\gamma_3\,\bar{q}_1(w)$ and $\Psi_1(w)=\gamma_6\,\bar{q}_2(w)+\gamma_9\,\bar{q}_3(w)$.

Noticeably, we gained a simple, all time domain relation between the original kernel and the components of the initial LSTV model: to the best of the authors' knowledge, such relationship was not derived before in literature.

Finally, in the case $\mathbb{P}=\mathbb{P}_0$, the system is a QILS: applying the above decomposition, the overall scheme reduces to a unique branch, simplifying the design complexity. Moreover, the down-sampler/up-sampler cascade performed at step 2) is no longer needed, so that $\gamma_a(\cdot)$ modulators disappear.

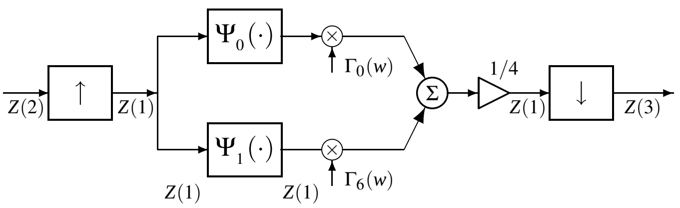

Fig. 6 – Equivalence with (mp,mq)–invariant architecture [5].

Differently from existing literature, our definition of time invariance applies naturally to multirate multidimensional systems. Conversely, the definition of (m,n)–shift invariance is limited to 1-dimensional systems. In particular, the proposed classification holds whenever the involved domains are Abelian groups. In the following we list some examples where the above framework can be applied in the multidimensional case.

The first example of application in the 2-dimensional case is the line interlace pattern used in television scanning [11][7]. In this context it is customary to apply up/downsampling between different and possibly non orthogonal 2-dimensional domains to obtain conversions between different TV standards: as previously shown, this kind of operation can be associated with QIL systems. Dealing with multidimensional systems, in [10] a new basic component, the exponential modulator, has been introduced. It allows to deal with the broad class of PIL systems, which ultimately is the more general class for multirate systems. For instance, exponential modulators were adopted in [12] to propose a multidimensional approach to Orthogonal Frequency Division Multiplexing (OFDM), starting from a basic architecture and obtaining efficient implementations. The schemes obtained in [12] fall in the general class of PILSs.

6. CONCLUSIONS

The concept of shift-invariance in multirate systems was presented in this paper following a fully time-domain approach, based on a group theoretic definition of signal domain [4][7]. The unified notation leads to a compact and effective description of invariance, which promises significant results in the analysis and synthesis of linear systems. Through a simple example, we showed that such definition brings an unambiguous and efficient decomposition and synthesis of multirate systems. Remarkably, the proposed approach applies to the time invariance of multirate multidimensional systems.

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