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ABOUT ORDERING SEQUENCES IN MINIMUM-PHASE SEQUENCES

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ABSTRACT

In the previous EUSIPCO 2004 paper, we have shown that certain types of sequences can be ordered such that the new sequence is a minimum-phase one. We also pointed out that this property is not valid for all real sequences. In this paper we prove that the set of points from the N-dimensional real or complex set having this characteristic is a non-empty open set. Moreover, we illustrate some new features of the sequences that can be ordered as minimum-phase sequences.

1. INTRODUCTION

During last decades there has been a great interest in developing special techniques for compression of data. For many types of archiving the amount of data needed for storage has been dramatically reduced. However, it may happen that for certain information the outcome is not so effective. It would be interested to design compression methods where the rate of compression is not very sensitive on the type of data. For instance, whether data belong from real or complex sets, only half of Fourier transform information should be needed.

One can find many signal processing applications which deal with signal reconstruction based on modulus or phase of the Fourier transform. In such situation the reconstruction of a complex sequence can be possible when we know in advance that its corresponding z-transform is a minimum-phase function or maximum-phase function [1]. The conventional reconstruction algorithms implies application of the Hilbert transform to the log-magnitude or phase of Fourier transform to provide the unknown component [2, 3]. An alternative approach consists in deriving iterative algorithms for reconstructing a minimum-phase or maximum-phase signal from the phase or magnitude of the Fourier transform [4].

Although clockwise or trigonometric order is mostly preferred [5], one can find applications dealing with sets of complex numbers where there is no preliminary request to pick the complex samples in clockwise or trigonometric order [6]. This means that one may select the succession of the complex samples, and the resulting sequence would be minimum-phase one. In this way the given set can be retrieved using only half of the information and can be used latter on for the primary goal [7]. It may happen also that any arrangement of a complex set into a sequence will not provide a minimum-phase sequence. Indeed, we have examined in [8] whether any finite set of real or complex numbers can be ordered such that the new corresponding complex sequence is a minimum-phase one and we have obtained that this property is not valid for all sequences.

The goal of this paper is to prove that the set having this characteristic is a non-empty open set in the N-dimensional real or complex set. Moreover, we shall add new sets of sequences that can be ordered as minimum-phase sequences. This paper is organized as follows. First we present the framework of our study (Section 2). Then we shall prove the main results mentioned above (Section 3). Finally we shall recall previous results and we relate them to our topic by some examples (Section 4).

2. FRAMEWORK

To proceed we need to specify some notations and representations. A sequence will be denoted by $\{x(n)\}_{n=\overline{0,M}}$ and the set will be designated by its usual symbol $\{x(0),x(1),\ldots,x(M)\}$. The n-th sample of the sequence will be written as $x(n)$. Also $(x(0),x(1),\ldots,x(M))$ will be a point from $\mathbb{R}^{M+1}$ or $\mathbb{C}^{M+1}$.

The z-transform of $\{x(n)\}_{n=\overline{0,M}}$ is:

$$X(z)=x(0)+x(1)z^{-1}+\cdots+x(M)z^{-M}=x(0)\prod_{p=1}^{M}(1-z_pz^{-1}), \tag{1}$$

where $z_p=r_pe^{j\theta_p}$, $p=1,2,\ldots,M$ are the zeros of $X(z)$.

The Fourier transform of $\{x(n)\}_{n=\overline{0,M}}$ is given by:

$$X(e^{j\omega})=X(z)|_{z=e^{j\omega}}=x(0)\prod_{p=1}^{M}(1-r_pe^{j\theta_p}e^{-j\omega}).$$

For $N=M+1$ the discrete Fourier transform of the given sequence $\{x(n)\}_{n=\overline{0,M}}$ is:

$$\begin{aligned}\widetilde{X}(k)&=X(z)|_{z=e^{j\frac{2\pi k}{N}}},\\ &=\left[x(0)+x(1)z^{-1}+\cdots+x(M)z^{-M}\right]|_{z=e^{j\frac{2\pi k}{N}}},\end{aligned}$$

where $k=\overline{0,N-1}$.

Since the length of the sequence is finite and $M+1=N$, $\widetilde{X}(k)$ are exactly the samples of the Fourier transform $X(e^{j\omega})$:

$$\widetilde{X}(k)=X(e^{j\omega})|_{\omega=\frac{2\pi k}{N}},\ k=0,1,\ldots,N-1,$$

and no frequency aliasing occurs when we reconstruct $X(e^{j\omega})$ from spectrum samples $\widetilde{X}(k)$ [9]. It follows that z-transform and Fourier transform can be found from the DFT samples:

$$X(z)=\sum_{n=0}^{N-1}x(n)z^{-n}=\frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1}\widetilde{X}(k)e^{\frac{j2\pi kn}{N}}\right]z^{-n}.$$

$$X(e^{j\omega})=\sum_{n=0}^{N-1}x(n)e^{-j\omega n}=\frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1}\widetilde{X}(k)e^{\frac{j2\pi kn}{N}}\right]e^{-j\omega n}.$$

Thus it is suggesting that a change in the way we pick the samples $x(n)$ (by modifying the succession) may affect the $X(z)$'s pole-zeros configuration, and consequently if $X(z)$ is minimum phase or non-minimum phase function or maximum-phase function. Nevertheless, our interest is to identify the minimum-phase sequences or maximum-phase sequences, since, in their case, from:

$$|\widetilde{X}(k)|=X(z)X(z^{-1})|_{z=e^{\frac{j2\pi k}{N}}},$$

the ambiguity of zero allocation is not anymore present [10]. In the following we shall focuss on minimum-phase sequences.

3. MAIN RESULTS

Definition 1. Let $\{x(0),x(1),\ldots,x(M)\}$ be a finite complex valued set. The set is said to have ordering minimum-phase (OMP) property if there exists a permutation

$$\begin{pmatrix} x(0) & x(1) & \ldots & x(M) \\ y(0) & y(1) & \ldots & y(M)\end{pmatrix}$$

of this set such that $Y(z)=y(0)+y(1)z^{-1}+\cdots+y(M)z^{-M}$ is a minimum-phase function.

The point $(x(0),x(1),\ldots,x(M))$ from $\mathbb{R}^{M+1}$ or $\mathbb{C}^{M+1}$ is said to have OMP property if the set $\{x(0),x(1),\ldots,x(M)\}$ has OMP property.

We have the main following theorem:

Theorem 1. *The set of all points from $\mathbb{R}^{M+1}$ (or $\mathbb{C}^{M+1}$) having OMP property is an open set.*

The proof of Theorem 1 is based on the well known theorem that the zeros of a polynomial function are continuous functions of the coefficients of the polynomial [11].

Theorem 2. *Let*

$$f(z)=a_0+a_1z+\cdots+a_nz^n=a_n\prod_{j=1}^{p}(z-z_j)^{m_j},\quad a_n\neq 0,$$

$$F(z)=(a_0+\varepsilon_0)+(a_1+\varepsilon_1)z+\cdots+$$

$$(a_{n-1}+\varepsilon_{n-1})z^{n-1}+a_nz^n$$

and let

$$0<r_k<\min|z_k-z_j|,\quad j=1,2,\ldots,k-1,k+1,\ldots,p. \quad (2)$$

There exists a positive number ε such that, if $|\varepsilon_i|\leq\varepsilon$ for $i=0,1,\ldots,n-1$, then $F(z)$ has precisely m_k zeros in the circle C_k with center at z_k and radius r_k.

Before proceeding to prove Theorem 1, we introduce two lemmas.

Lemma 1. *Let*

$$Y_1(z)=y(0)+y(1)z^{-1}+\cdots+y(M-1)z^{-(M-1)}+y(M)z^{-M}$$

$$=y(0)\prod_{j=1}^{p'}(1-z'_jz^{-1})^{m'_j},\quad y(0)\neq 0,$$

$$Y_2(z)=[y(0)+\delta_0]+[y(1)+\delta_1]z^{-1}+\cdots+$$

$$[y(M-1)+\delta_{M-1}]z^{-(M-1)}+[y(M)+\delta_M]z^{-M},$$

with $0<|\delta_0|<|y(0)|$, and let

$$0<r'_k<\min|z'_k-z'_j|,\quad j=1,2,\ldots,k-1,k+1,\ldots,p'. \quad (3)$$

There exists a positive number δ such that, if $|\delta_i|\leq\delta$ for $i=0,1,\ldots,M$, then $Y_2(z)$ has precisely m'_k zeros in the circle C'_k with center at z'_k and radius r'_k.

Proof: In Theorem 2 we replace

- n by M;
- a_n by $y(0)$;
- a_k by $y(M-k)/y(0)$

and ε_k by

$$\frac{\delta_{M-k}y(0)-\delta_0y(M-k)}{y(0)[y(0)+\delta_0]}.$$

Let denote by z'_j, p' and r'_k the corresponding values for z_j, p and r_k. There exists a positive number ε such that, if $|\varepsilon_i|\leq\varepsilon$ for $i=0,1,\ldots,n-1$, then $F(z)$ has precisely m'_k zeros in the circle C'_k with center at z'_k and radius r'_k. But

$$Y_2(z)=\frac{y(0)+\delta_0}{z^M}F(z)$$

will have also precisely m'_k zeros in the circle C'_k with center at z'_k and radius r'_k whenever

$$\left|\frac{\delta_{M-k}y(0)-\delta_0y(M-k)}{y(0)[y(0)+\delta_0]}\right|\leq\varepsilon$$

is satisfied. Indeed, there exists

$$\delta=\varepsilon\,\frac{[|y(0)|-|\delta_0|]|y(0)|}{|y(0)|+\max|y(k)|}>0$$

such that if $\delta_k\leq\delta$, then

$$\left|\frac{\delta_{M-k}y(0)-\delta_0y(M-k)}{y(0)[y(0)+\delta_0]}\right|\leq\frac{|\delta|[|y(0)|+|y(M-k)|]}{|y(0)[y(0)+\delta_0]|}$$

$$=\varepsilon\frac{|y(0)|[|y(0)|-|\delta_0|]}{|y(0)|+\max|y(k)|}\,\frac{|y(0)|+|y(M-k)|}{|y(0)+\delta_0||y(0)|}\leq\varepsilon$$

which satisfies the above mentioned condition.

□

Lemma 2. *Consider the point $(y(0),y(1),\ldots,y(M))$ from $\mathbb{R}^{M+1}$ (or $\mathbb{C}^{M+1}$) such that*

$$Y(z)=y(0)+y(1)z^{-1}+\cdots+y(M)z^{-M}$$

is a minimum-phase function. Then there exists a neighborhood U of the point $(y(0),y(1),\ldots,y(M))$ and any point $(y'(0),y'(1),\ldots,y'(M))\in U$ provides a minimum-phase function

$$Y'(z)=y'(0)+y'(1)z^{-1}+\cdots+y'(M)z^{-M}.$$

Proof: If $Y(z)$ is a minimum-phase function, then all zeros are inside the unit circle. To prove our result, we just apply Lemma 1, by selecting

$$r'_k = \min\{|z'_k - z'_j|, 0.99\min_{|z|=1}|z - z'_k|\}, \quad j = 1,2,\ldots,k-1,k+1,\ldots,p'. \tag{4}$$

Then there exists a positive number δ such that, if $|\delta_i| \le \delta$ for $i = 0,1,\ldots,M$, then

$$Y_2(z) = [y(0)+\delta_0] + [y(1)+\delta_1]z^{-1} + \cdots +$$

$$[y(M-1)+\delta_{M-1}]z^{-(M-1)} + [y(M)+\delta_M]z^{-M} \equiv Y'(z),$$

has precisely m'_k zeros in the circle C'_k with center at z'_k and radius r'_k. We further note from (4), that all zeros of $Y'(z)$ are inside the unit circle. Also $(y'(0), y'(1), \ldots, y'(M)) \in U$, where U is the closed ball with center $(y(0), y(1), \ldots, y(M))$ and radius δ.

□

Now we shall prove Theorem 1.

For any point $(x(0), x(1), \ldots, x(M))$ from $\mathbb{R}^{M+1}$ (or $\mathbb{C}^{M+1}$) having OMP property, there exists a permutation

$$\mathscr{P} = \begin{pmatrix} x(0) & x(1) & \ldots & x(M) \\ y(0) & y(1) & \ldots & y(M) \end{pmatrix}$$

such that $Y(z) = y(0) + y(1)z^{-1} + \cdots + y(M)z^{-M}$ is a minimum-phase function. Using Lemma 2, there should be an entire neighborhood U of the point $(y(0), y(1), \ldots, y(M))$ such that any point $(y'(0), y'(1), \ldots, y'(M)) \in U$ provides a minimum-phase function $Y'(z) = y'(0) + y'(1)z^{-1} + \cdots + y'(M)z^{-M}$.

Consider now the set $V = \mathscr{P}^{-1}(U)$. It is an open set, as preimage of an open set through a bijection. Since $(y(0), y(1), \ldots, y(M)) \in U$, then V is an neighborhood of $(x(0), x(1), \ldots, x(M))$. Any point $(x'(0), x'(1), \ldots, x'(M)) \in V$ has OMP property as there exist the point $(y'(0), y'(1), \ldots, y'(M)) \in U$ and the permutation

$$\mathscr{P} = \begin{pmatrix} x'(0) & x'(1) & \ldots & x'(M) \\ y'(0) & y'(1) & \ldots & y'(M) \end{pmatrix}$$

such that $Y'(z) = y'(0) + y'(1)z^{-1} + \cdots + y'(M)z^{-M}$ is a minimum-phase function.

Thus we have proven that the set of all points from $\mathbb{R}^{M+1}$ (or $\mathbb{C}^{M+1}$) having OMP property is an open set.

Remark 1 . In our proof, we need to have $y(0) \neq 0$ for the minimum-phase function $Y(z)$. We just note that if $y(0) = 0$, then $Y(\infty) = 0$, thus $Y(z)$ is not a minimum-phase function.

Other useful results will come in the sequel.

Proposition 1 . *The set of all points from $\mathbb{R}^{M+1}$ (or $\mathbb{C}^{M+1}$) having OMP property is non empty.*

Proof: Let a be such that $|a| < 1$ and consider the set:

$$\Big\{1,\ \tfrac{M!}{1!(M-1)!}a,\ \tfrac{M!}{2!(M-2)!}a^2,\ \ldots,$$

$$+\tfrac{M!}{k!(M-k)!}a^k,\ \ldots,\ \tfrac{M!}{M!1!}a^M\Big\}.$$

Then

$$X(z) = 1 + \tfrac{M!}{1!(M-1)!}az^{-1} + \tfrac{M!}{2!(M-2)!}a^2z^{-2} + \ldots$$

$$+\tfrac{M!}{k!(M-k)!}a^kz^{-k} + \ldots + \tfrac{M!}{M!1!}a^Mz^{-M} = (1+az^{-1})^M$$

has M zeros inside the unit circle.

□

Conversely, its complement (the set who has not OMP property) is closed and may be empty. In view of next proposition, this is not the case.

Proposition 2 . *Whenever $x(0) + x(1) + \cdots + x(M) = 0$, the set $\{x(n) | n = \overline{0,M}\}$ has not OMP property.*

Proof: Indeed, in this situation, for any permutation we have $Y(1) = 0$. In such case $\{y(n)\}_{n=\overline{0,M}}$ is a non-minimum phase sequence. This is always possible if the center of the gravity of the set $\{x(0), x(1), \ldots, x(M)\}$ is selected as the origin of the axes.

□

Remark 2 .

a. Sometimes we can have $x(0) - x(1) + \cdots + (-1)^M x(M) = 0$, and the set $\{x(n) | n = \overline{0,M}\}$ has OMP property. This can be possible, for instance if $\{x(0), x(1), x(2)\} = \{-1/4, 3/4, 1\}$ and $y(0) = 1$, $y(1) = -1/4$, $y(2) = 3/4$.

b. Any set $\{x(0), x(1)\}$ with $|x(0)| \neq |x(1)|$ has OMP property. Indeed, $x(0) + x(1)z^{-1}$ and $x(1) + x(0)z^{-1}$ should have reciprocal zeros, located inside and outside the unit circle.

c. If $0 \in \{x(0), x(1), \ldots, x(M)\}$, there may appear some ambiguities about the number of zeros inside the unit circle, especially when, after permutation, the leading coefficient is zero. However, if $Y(z) = y(0) + y(1)z^{-1} + \cdots + y(M-1)z^{-M+1} + 0z^{-M}$ has $M-1$ zeros inside unit circle, then it is a minimum-phase function. Consequently, in such case $\{x(0), x(1), \ldots, x(M)\}$ has OMP property.

4. EXAMPLES OF ORDERING SEQUENCES INTO MINIMUM-PHASE SEQUENCES

Previous work has shown that [8]:

Proposition 3 . *Any set of real, positive and distinct numbers $\{x(0), x(1), \ldots, x(M)\}$ has OMP property.*

Proposition 4 . *For any set of real numbers $\{x(i) | i = \overline{0,2}\}$, there is a choice of ordering them such that the corresponding new sequence is a minimum-phase one.*

Similar statements as Proposition 4 cannot be found so easily for other values of $M > 2$. For $M = 3$ it can be shown that the answer is positive only in special cases. Actually we have proven:

Proposition 5 . *For any set of real numbers $\{x(i) | i = \overline{0,3}\}$, which differ in modulus and satisfying both $x(0)x(1)x(2)x(3) > 0$ and $x(0) + x(1) + x(2) + x(3) \neq 0$, there is a choice of ordering them such that the corresponding new sequence is a minimum-phase one.*

Simulations have shown that for $M = 3$ we can find situations that no ordering of real sequences will produce a minimum-phase sequence. One example is presented in [8]. Our simulations also verify that the sequences where any kind of ordering fails consists of samples with an odd number of plus and minus signs. In such situation, a special case appears when the sum of the numbers is zero, when the zeros may lie outside the unit disk or on the unit circle. It follows also that for $M \geq 3$ one can find sequences that no ordering will provide a minimum-phase sequence. Indeed, we have numerically verified that for every $M = 4$ to 9, the set $\{-1,-2,-3,\dots,-M,\ M+1\}$ has not OMP property. Also from $M = 10$ to 12, we have performed random permutations of the set $\{-1,-2,-3,\dots,-M,\ M+1\}$, and no minimum-phase sequence has been detected.

One may ask whether any set of complex numbers has OMP property. The answer is negative (Table 1). Simulations and detailed analysis can show that for characterizing sets of three complex numbers we should consider not only their modulus, but also the localization of one complex number with respect to the interior/exterior bisector of the other two. However, we can determine a certain case when three complex numbers can be ordered as minimum-phase sequence.

Lemma 3 . *Consider the second order complex polynomial* $P(z) = a_0 + a_1 z^{-1} + a_2 z^{-2}$.

1. *If* $|a_2| \geq |a_0|$, *then* $P(z)$ *is non minimum-phase function.*
2. *If* $|a_1|+|a_2| < |a_0|$, *then* $P(z)$ *is minimum-phase function.*
3. *If* $|a_2| < |a_0|$ *and* $|a_1| > |a_0|+|a_2|$, *then* $P(z)$ *is non minimum-phase function.*

Proof: For the beginning let us focus on the Schur-Cohn conditions for $P(z)$ (Appendix A.2) and let us consider $a_1/a_0 = \rho_1 e^{j\theta_1}$ and $a_2/a_0 = \rho_2 e^{j\theta_2}$, where $\rho_i > 0$, $i = 1,2$. The Schur-Cohn conditions can be written as follows:

$$|a_2| < |a_0|; \quad |\overline{a}_0 a_1 - a_2 \overline{a}_1| < |a_0|^2 - |a_2|^2$$

or

$$\rho_2 < 1; \quad \left|\rho_1 e^{j\theta_1} - \rho_1 \rho_2 e^{j(\theta_2-\theta_1)}\right| < 1-\rho_2^2. \tag{5}$$

The last inequality is valid whenever:

$$\cos(2\theta_1 - \theta_2) > \frac{1+\rho_2^2 - \left(\frac{1-\rho_2^2}{\rho_1}\right)^2}{2\rho_2}. \tag{6}$$

Let us denote by

$$E(\rho_1,\rho_2) = \frac{1+\rho_2^2 - \left(\frac{1-\rho_2^2}{\rho_1}\right)^2}{2\rho_2}.$$

We have

$$E(\rho_1,\rho_2)+1 = \tfrac{(1+\rho_2)^2}{2\rho_1^2\rho_2}[\rho_1^2 - (1-\rho_2)^2] =$$

$$\tfrac{(1+\rho_2)^2}{2\rho_1^2\rho_2}(\rho_1+\rho_2-1)(\rho_1-\rho_2+1);$$

$$E(\rho_1,\rho_2)-1 = \tfrac{(1+\rho_2)^2}{2\rho_1^2\rho_2}[\rho_1^2 - (1+\rho_2)^2] =$$

$$\tfrac{(1-\rho_2)^2}{2\rho_1^2\rho_2}(\rho_1+\rho_2+1)(\rho_1-\rho_2-1).$$

Consequently, if $\rho_2 < 1$,

- whenever $\rho_1+\rho_2-1 < 0$, then $E(\rho_1,\rho_2) < -1$;
- whenever $\rho_1-\rho_2-1 > 0$, then $E(\rho_1,\rho_2) > 1$.

1. If $|a_2| \geq |a_0|$, then first condition from (5) is not satisfied. It follows that $P(z)$ is non minimum-phase function.
2. If $|a_1|+|a_2| < |a_0|$, then $|a_2| < |a_0|$ also. It follows that $\rho_2 < 1$ and $\rho_1+\rho_2 < 1$. For any θ_1, θ_2, relationship (6) is satisfied. Thus $P(z)$ is minimum-phase function.
3. If $|a_2| < |a_0|$ and $|a_1| > |a_0|+|a_2|$, then $\rho_2 < 1$ and $\rho_1 > 1+\rho_2$. For any θ_1, θ_2, relationship (6) is not satisfied. Thus $P(z)$ is non minimum-phase function.

This ends the proof of Lemma 3.

□

Proposition 6 . *Any set of complex numbers* $\{x(0),x(1),x(2)\}$, *which satisfies*

$$\max\{|x(0)|,|x(1)|,|x(2)|\} > \text{median}\{|x(0)|,|x(1)|,|x(2)|\} + \min\{|x(0)|,|x(1)|,|x(2)|\} \tag{7}$$

has OMP property.

Proof: To prove Proposition 6, we start by assuming that the set has been ordered in $\{y(0),y(1),y(2)\}$ such that:

$$|y(0)| > \max\{|y(1)|,|y(2)|\},$$

thus the condition from the statement of Proposition 6 is $|y(0)| > |y(1)|+|y(2)|$. Besides $|y(2)| < |y(0)|$ and we retrieve Case 2) of Lemma 3. Consequently $Y(z)$ is minimum-phase function.

Thus any set of complex numbers within statement conditions, has OMP property.

□

The second statement of Lemma 3 is a special case of Cohn's theorem ([11], pp. 130), reformulated here as follows:

Theorem 3 . *If* $|a_0| > |a_n|+|a_{n-1}|+\dots+|a_1|$, *then* $a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}$ *has exactly n zeros.*

This result can provide us real or complex sets having OMP property of any length. This development is similar with Proposition 6.

One of the properties of minimum-phase systems [12] is the theorem concerning energy conservation (Appendix B). It should be noted that by reordering we keep the energy of the sequence, however the magnitude of the Fourier transform can be easily changed. Thus the condition for a certain sequence $\{x(n)\}_{n=\overline{0,M}}$ to have its energy concentrated around origin such that:

$$|x(0)| > |x(1)| > \cdots > |x(M)| > 0 \tag{8}$$

and the property to be minimum-phase sequence are not equivalent. However for real positive sequences the equivalence holds, as a consequence of Eneström-Kakeya theorem [13].

To conclude this section, we have proven that there are cases when a set can be ordered into a minimum-phase sequence. However, we cannot guarantee this property for all real or complex sequences. Moreover, in addition to the situations mentioned by Proposition 2, one can find sets without OMP property when $M \geq 2$ for complex sequences and if $M \geq 3$ for real sequences.

$y(0)$	$y(1)$	$y(2)$	z_1	z_2	$\|z_1\|$	$\|z_2\|$
3+2j	3+3.5j	4+2j	-0.8068 + 1.1315j	-0.4240 - 0.7854j	1.3897	0.8925
3+2j	4+2j	3+3.5j	-0.8463 - 1.0315j	-0.3845 + 0.8777j	1.3343	0.9582
3+3.5j	3+2j	4+2j	-0.2097 - 0.9955j	-0.5433 + 0.7837j	1.0173	0.9536
3+3.5j	4+2j	3+2j	-0.4188 - 0.9559j	-0.4754 + 0.5794j	1.0436	0.7495
4+2j	3+2j	3+3.5j	-0.5974 - 0.8618j	-0.2026 + 0.9618j	1.0487	0.9830
4+2j	3+3.5j	3+2j	-0.5323 + 0.9859j	-0.4177 - 0.5859j	1.1204	0.7196

Table 1: The set $\{x(0),x(1),x(2)\} = \{3+2j, 3+3.5j, 4+2j\}$ has not OMP property.

5. CONCLUSIONS

In this paper we have focussed on the issue of ordering of a sequence into a minimum-phase sequence. We have proven that the set of points from the N-dimensional real or complex set having this characteristic is a non-empty open set. Moreover, we clarify some features of the sequences that can be ordered as minimum-phase sequences.

Related issues deserve to be investigated. Using Schur-Con recursions and all possible permutations, one can find if a set has OMP property and can also find the corresponding minimum-phase sequence, derived from the given sequence. However, this may be computational expensive and other fast methods would be highly appreciated.

A. SCHUR-COHN STABILITY TEST

A.1 Schur-Cohn recursion

Let $A_M(z)$ be a complex polynomial of order M in z^{-1}:

$$A_M(z) = \alpha_M(0) + \alpha_M(1)z^{-1} + \cdots + \alpha_M(M)z^{-M},\ \alpha_M(0) = 1.$$

All the zeros of $A_M(z)$ lie inside the unit circle if and only if $|k_m| < 1$, for $m = M, M-1, \cdots, 1$, where [14]:

$$k_m = \alpha_m(m);\ B_m(z) = z^{-m}A_m^*(z^{-1}),$$

$$A_{m-1}(z) = \frac{A_m(z) - k_m B_m(z)}{1 - |k_m|^2}.$$

A.2 Second order polynomial

For a second order complex polynomial in z^{-1}, the Schur-Cohn recursion is the following one:

$$k_2 = \alpha_2(2);\ B_2(z) = \alpha_2^*(2) + \alpha_2^*(1)z^{-1} + \alpha_2(0)z^{-2},$$

$$A_1(z) = \frac{A_2(z) - k_2 B_2(z)}{1 - |k_2|^2} = 1 - \frac{\alpha_2(1) - \alpha_2(2)\alpha_2^*(1)}{1 - |\alpha_2(2)|^2} z^{-1},$$

and all the zeros lie inside the closed unit disk if and only if

$$|\alpha_2(2)| < 1; \quad \left| \frac{\alpha_2(1) - \alpha_2(2)\alpha_2^*(1)}{1 - |\alpha_2(2)|^2} \right| < 1. \tag{9}$$

B. ENERGY CONCENTRATION THEOREM

Theorem 4 . *If the systems $H(z)$ (non-minimum phase) and $H_m(z)$ (minimum-phase) have the same magnitude response and their response to the same input are $g(n)$ and $y(n)$, respectively, then for any n_0,*

$$\sum_{n=0}^{n_0} |y(n)|^2 \geq \sum_{n=0}^{n_0} |g(n)|^2. \tag{10}$$

REFERENCES

[1] L. A. Zadeh and C. A. Desoer, *Linear system theory*, Mc Graw Hill, New York, 1969.

[2] M. H. Hayes, J. S. Lim, and A. V. Oppenheim, “Signal reconstruction from phase or magnitude,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-28, no. 6, pp. 672–680, Dec. 1980.

[3] B. D. O. Anderson and M. Green, “Hilbert transform and gain/phase error bounds for rational functions,” *IEEE Transactions on Circuits and Systems*, vol. 35, no. 5, pp. 528–535, May 1988.

[4] Thomas F. Quatieri and A. V. Oppenheim, “Iterative techniques for minimum phase signal reconstruction from phase or magnitude,” *IEEE Trans. ASSP*, vol. ASSP-29, no. 6, pp. 1187–1193, Dec. 1981.

[5] A. K. Jain, *Fundamentals of Image Processing*, Prentice Hall International, Inc., Englewood Cliffs, NJ 07632, 1989.

[6] B. Kingslake and Brian J. Thompson, Eds., *Applied Optics and Optical Engineering*, Academic Press, New-York, 1980.

[7] Corneliu Rusu and Pauli Kuosmanen, “Phase approximation by logarithmic sampling of gain,” *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 50, no. 2, pp. 93–101, Feb. 2003.

[8] Corneliu Rusu and Jaakko Astola, “Sequences ordering in minimum-phase sequences,” in *Proc. EUSIPCO 2004*, TU Wien, Sept. 2004, pp. 93–96.

[9] J. G. Proakis and D. G. Manolakis, *Introduction to digital signal processing*, Macmillan, New-York, 1988.

[10] S. K. Mitra, *Digital signal processing: a computer-based approach*, McGraw-Hill, New-York, 1998.

[11] Morris Marden, *Geometry of polynomials*, American Mathematical Society, Providence, Rhode Island, 1989.

[12] A. Papoulis, *Signal Analysis*, McGraw-Hill Book Company, New York, 1977.

[13] S. D. Fisher, *Complex Analysis*, Wadsworth & Books, Pacific Grove, California, 2nd edition, 1990.

[14] D. J. DeFatta, J. G. Lucas, and W. S. Hodgkiss, *Digital Signal Processing: A System Design Approach*, John Wiley & Sons, New-York, 1988.