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NONPARAMETRIC METHOD FOR DETECTING THE NUMBER OF NARROWBAND SIGNALS WITHOUT EIGENDECOMPOSITION IN ARRAY PROCESSING

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ABSTRACT

A computational simple and efficient nonparametric method for estimating the number of signals without eigendecomposition (MENSE) is proposed for the narrowband signals impinging on a uniform linear array (ULA). When finite array data are available, a new detection criterion is formulated in terms of the row elements of the QR upper-triangular factor of the auto-product of a matrix formed from the cross-correlations between some sensor data. Then the number of signals is determined as a value for which this ratio criterion is maximized, where the QR decomposition with column pivoting is also used to improve detection performance. The proposed estimator is asymptotically consistent, and it is superior in detecting closely-spaced signals with a small number of snapshots and/or at low signal-to-noise ratio (SNR).

1. INTRODUCTION

Estimating the number of incident signals from noisy array data is an essential prerequisite for high-resolution direction-of-arrival (DOA) estimation in array processing (e.g., [1], [2] and references therein), where the performance of direction estimation is adversely affected if the number of signals is inaccurately determined. When the incoming signals are noncoherent in the presence of temporally and spatially white Gaussian additive noise, the number of signals can be determined from the “multiplicity” of the smallest eigenvalues of the array covariance matrix. Consequently, eigenstructure-based nonparametric detection methods have been proposed and attained considerable prominence because of their relatively computational simplicity without the need to estimate direction parameters, and the most popular ones are the Akaike information criterion (AIC) and the minimum description length (MDL) criterion [3], which are formulated in terms of eigenvalues. However, these nonparametric methods suffer from serious degradation, when the incident signals are coherent (i.e., fully correlated) such as in multipath propagation environments, where the rank of the source signal covariance matrix is smaller than the number of signals.

Although nonparametric methods such as the AIC and MDL criterion can be modified to combat the deleterious effect of coherency between the incident signals by using decorrelation techniques such as (forward-backward (FB)) spatial smoothing (SS) [4], [5], their performance is usually susceptible to the accuracy of the estimated eigenvalues. Furthermore, most of the aforementioned nonparametric methods require the eigendecomposition of the (smoothed) correlation matrix, and thus their applications are limited in some applications, because the eigendecomposition process is not suitable for real-time implementation due to its computational burdensomeness and time-consuming (e.g., [6]). Thus a considerable amount of computation required for eigendecomposition turns out to be a major obstacle to real-time implementation of most detection methods, especially when the number of sensors is large and/or online detection is required. Since the QR decomposition requires much lesser computational effort, some QR-based detection methods were proposed (e.g., [7], [8]). However, they needs *a priori* knowledge of true noise variance and subjective assessment or perform poorly in difficult scenarios.

We proposed a new QR-based detection method for the coherent signals impinging on a uniform linear array (ULA) [9]. By exploiting the array geometry and its shift invariance property to decorrelate the coherency of signals through subarray averaging and to eliminate the noise effect, the number of signals is revealed in the rank of the QR upper-trapezoidal factor of the auto-product of a combined Hankel matrix formed from the cross-correlations between some sensor data. When finite array data are available, a detection criterion was formulated in terms of the row elements of the QR upper-triangular factor, and the number of signals is determined as a value for which this ratio criterion is maximized. Unfortunately, the problem of detecting the absence of signals and the statistical analysis were not studied therein.

Therefore in this paper, we further strengthen the proposed method [9] and analyze its statistical property, where the absence or presence of incident signal(s) is detected by quantitatively comparing one element of the QR factor matrix with an auto-correlation of array data without the need for any “manual” adjustment. The statistical analysis clarifies that the proposed estimator is asymptotically consistent and its detection performance can be predicted by examining the QR decomposition of the asymptotical auto-product matrix with different permutation matrices, where the choice of the predetermined QR permutation matrix is also considered. Because the EVD/SVD and the evaluation of all correlations of array data are not needed, the proposed method is computationally efficient and suitable for real-time implementation having remarkable insensitivity to the correlation of incident signals. The effectiveness of proposed method is verified through numerical examples.

2. DATA MODEL AND BASIC ASSUMPTIONS

We consider a ULA of M sensors with spacing d and suppose that p ($p < M/2$) narrowband signals $\{s_k(n)\}$ with the carrier frequency f_0 are far away and impinge on the array from distinct directions $\{\theta_k\}$. The received signal $y_i(n)$ at the ith sensor can be expressed

$$y_i(n) = \sum_{k=1}^{p} s_k(n)\, e^{j\omega_0 (i-1)\tau(\theta_k)} + w_i(n) \tag{1}$$

where $w_i(n)$ is the additive noise, $\omega_0 \triangleq 2\pi f_0$, $\tau(\theta_k) \triangleq (d/c)\sin\theta_k$, and c is the propagation speed frequency. Then the received signals can be rewritten in a compact form as

$$\boldsymbol{y}(n) = \boldsymbol{A}\boldsymbol{s}(n) + \boldsymbol{w}(n) \tag{2}$$

where $\boldsymbol{y}(n)$, $\boldsymbol{s}(n)$, and $\boldsymbol{w}(n)$ are the vectors of the received signals, incident signals, and additive noise, and $\boldsymbol{A}$ is the array response matrix given by $\boldsymbol{A} \triangleq [\boldsymbol{a}(\theta_1), \cdots, \boldsymbol{a}(\theta_p)]$ with $\boldsymbol{a}(\theta_k) \triangleq [1, e^{j\omega_0\tau(\theta_k)}, \cdots, e^{j\omega_0(M-1)\tau(\theta_k)}]^T$.

Here we assume that the array is calibrated and the matrix $\boldsymbol{A}$ is unambiguous. The signals $\{s_k(n)\}$ are assumed to be coherent under the flat-fading multipath propagation and given by [10]

$$s_k(n) = \beta_k s_1(n), \qquad \text{for} \quad k = 1, 2, \cdots, p \tag{3}$$

where $\{\beta_k\}$ are the complex attenuation coefficients with $\beta_k \neq 0$ and $\beta_1 = 1$. The incident signals and additive noise are assumed to be independent and complex circularly Gaussian noise with zero-mean and variance as $E\{s_1(n) \cdot s_1^*(t)\} = c_s\delta_{n,t}$, $E\{s_1(n)s_1(t)\} = 0$, $E\{\boldsymbol{w}(n)\boldsymbol{w}^H(t)\} = \sigma^2\boldsymbol{I}_M\delta_{n,t}$, and $E\{\boldsymbol{w}(n)\boldsymbol{w}^T(t)\} = \boldsymbol{O}_{M\times M}$ $\forall n, t$, where $E\{\cdot\}$, $(\cdot)^H$, $\delta_{n,t}$, $\boldsymbol{I}_m$, and $\boldsymbol{O}_{m\times m}$ denote the expectation, Hermitian transpose, Kronecker delta, and $m \times m$ identity and null matrices.

3. METHOD FOR ESTIMATING THE NUMBER OF SIGNALS WITHOUT EIGENDECOMPOSITION (MENSE)

3.1. QR-Based Detection Method

By dividing the full array into L overlapping subarrays with $\bar{p}$ sensors in the forward and backward directions [4], [5], the signal vectors of the lth forward and backward subarrays are given by $\boldsymbol{y}_{fl}(n) \triangleq [y_l(n), y_{l+1}(n), \cdots, y_{l+\bar{p}-1}(n)]^T$, and $\boldsymbol{y}_{bl}(n) \triangleq [y_{M-l+1}(n), y_{M-l}(n), \cdots, y_{L-l+1}(n)]^H$ for $l = 1, 2, \cdots, L$, where $L = M - \bar{p} + 1$, and $\bar{p} \geq p$. By defining four correlation vectors $\boldsymbol{\varphi}_{fl}$, $\bar{\boldsymbol{\varphi}}_{fl}$, $\boldsymbol{\varphi}_{bl}$, and $\bar{\boldsymbol{\varphi}}_{bl}$ between these vectors $\boldsymbol{y}_{fl}(n)$ and $\boldsymbol{y}_{bl}(n)$ and the signals $y_1(n)$ and $y_M(n)$ as $\boldsymbol{\varphi}_{fl} \triangleq E\{\boldsymbol{y}_{fl}(n)y_M^*(n)\}$, $\bar{\boldsymbol{\varphi}}_{fl} \triangleq E\{\boldsymbol{y}_{fl}(n)y_1^*(n)\}$, $\boldsymbol{\varphi}_{bl} \triangleq E\{y_1(n)\boldsymbol{y}_{bl}(n)\}$, and $\bar{\boldsymbol{\varphi}}_{bl} \triangleq E\{y_M(n)\boldsymbol{y}_{bl}(n)\}$, we can get four $(M - \bar{p}) \times \bar{p}$ Hankel correlation matrices

$$\boldsymbol{\Phi}_f = [\boldsymbol{\varphi}_{f1}, \cdots, \boldsymbol{\varphi}_{fL-1}]^T, \quad \bar{\boldsymbol{\Phi}}_f = [\bar{\boldsymbol{\varphi}}_{f2}, \cdots, \bar{\boldsymbol{\varphi}}_{fL}]^T \tag{4}$$

$$\boldsymbol{\Phi}_b = [\boldsymbol{\varphi}_{b1}, \cdots, \boldsymbol{\varphi}_{bL-1}]^T, \quad \bar{\boldsymbol{\Phi}}_b = [\bar{\boldsymbol{\varphi}}_{b2}, \cdots, \bar{\boldsymbol{\varphi}}_{bL}]^T. \tag{5}$$

Then by defining an $(M - \bar{p}) \times 4\bar{p}$ correlation matrix $\boldsymbol{\Phi}$ as $\boldsymbol{\Phi} \triangleq [\boldsymbol{\Phi}_f, \bar{\boldsymbol{\Phi}}_f, \boldsymbol{\Phi}_b, \bar{\boldsymbol{\Phi}}_b]$, after some algebraic manipulations, we obtain an auto-product $\boldsymbol{\Psi}$ of matrix $\boldsymbol{\Phi}$ as [10], [11]

$$\boldsymbol{\Psi} = \boldsymbol{\Phi}\boldsymbol{\Phi}^H = c_s^2\,|\bar{\rho}_M|^2\,\bar{\boldsymbol{A}}\boldsymbol{B}\boldsymbol{F}\boldsymbol{F}^H\boldsymbol{B}^H\bar{\boldsymbol{A}}^H \tag{6}$$

where $\boldsymbol{B} = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_p)$, $\boldsymbol{F} = [\bar{\boldsymbol{A}}_1^T, (\bar{\rho}_1/\bar{\rho}_M)\boldsymbol{D} \cdot \bar{\boldsymbol{A}}_1^T, (\bar{\rho}_1^*/\bar{\rho}_M)\bar{\boldsymbol{B}}\boldsymbol{D}^{-(M-1)}\bar{\boldsymbol{A}}_1^T, (\bar{\rho}_M^*/\bar{\rho}_M)\bar{\boldsymbol{B}}\boldsymbol{D}^{-(M-2)}\bar{\boldsymbol{A}}_1^T]$, $\boldsymbol{D} = \mathrm{diag}(e^{j\omega_0\tau(\theta_1)}, e^{j\omega_0\tau(\theta_2)}, \cdots, e^{j\omega_0\tau(\theta_p)})$, $\bar{\boldsymbol{B}} = \boldsymbol{B}^{-1} \cdot \boldsymbol{B}^*$, $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{A}}_1$ are the submatrices of the matrix $\boldsymbol{A}$ consisting of its first $M - \bar{p}$ and $\bar{p}$ rows, $\bar{\rho}_i = \boldsymbol{\beta}^H\boldsymbol{b}_i^*(\theta)$, $\boldsymbol{\beta} = [\beta_1, \beta_2, \cdots, \beta_p]^T$, and $\boldsymbol{b}_i(\theta) = [e^{j\omega_0(i-1)\tau(\theta_1)}, e^{j\omega_0(i-1)\tau(\theta_2)}, \cdots, e^{j\omega_0(i-1)\tau(\theta_p)}]^T$.

Clearly the number of signals p equals the rank of $\boldsymbol{\Psi}$ irrespective of the signal coherency iff the detectability condition that $p \leq \bar{p} < M - p$ is satisfied ($p \geq 1$) and is revealed in the rank of the QR upper-trapezoidal factor $\boldsymbol{R}$ of $\boldsymbol{\Psi}$ given by

$$\begin{aligned}\boldsymbol{\Psi} &= \boldsymbol{Q}\boldsymbol{R} \\ &= \boldsymbol{Q}\begin{bmatrix} \boldsymbol{R}_{11}, & \boldsymbol{R}_{12} \\ \multicolumn{2}{c}{\boldsymbol{O}_{(M-\bar{p}-p)\times(M-\bar{p})}} \end{bmatrix}\begin{matrix}\}p \\ \}M-\bar{p}-p\end{matrix}\end{aligned} \tag{7}$$

where $\boldsymbol{Q}$ is the $(M - \bar{p}) \times (M - \bar{p})$ unitary matrix, $\boldsymbol{R}_{11}$ is the $p \times p$ upper-triangular and nonsingular matrix, and $\boldsymbol{R}_{12}$ is the $p \times (M - \bar{p} - p)$ matrix with non-zero elements.

Proof: Omitted (see [11] for details). ∎

Remark A: When the incoming signals are uncorrelated, we can get four Hankel correlation matrices as

$$\boldsymbol{\Phi}_f = \boldsymbol{\Phi}_b = \bar{\boldsymbol{A}}\boldsymbol{D}^{-(M-1)}\boldsymbol{C}_s\bar{\boldsymbol{A}}_1^T \tag{8}$$

$$\bar{\boldsymbol{\Phi}}_f = \bar{\boldsymbol{\Phi}}_b = \bar{\boldsymbol{A}}\boldsymbol{D}\boldsymbol{C}_s\bar{\boldsymbol{A}}_1^T \tag{9}$$

where $\boldsymbol{C}_s \triangleq E\{\boldsymbol{s}(n)\boldsymbol{s}^H(n)\} = \mathrm{diag}(c_{s_1}, c_{s_2}, \cdots, c_{s_p})$, and $c_{s_k} \triangleq E\{s_k(n)s_k^*(n)\}$. Clearly the ranks of these matrices equal the number of incident signals p, and the structure of QR decomposition in (7) are still valid. □

3.2. Detecting Absence of Incident Signal

In the detection problem of array processing, we sometimes encounter the absence of incident signal(s) (i.e., $p = 0$). For the special case of a single signal (i.e., $p = 1$), the analytical expressions of the QR upper-trapezoidal factor $\boldsymbol{R}$ of the auto-product $\boldsymbol{\Psi}$ and the auto-correlation c_{ii} of the received signal $y_i(n)$ can be explicitly given by

$$\boldsymbol{R} = -4c_s^2\bar{p}(M-\bar{p})^{1/2} \cdot \begin{bmatrix} 1, & e^{-j\omega_0\tau(\theta_1)}, & \cdots & e^{-j\omega_0(L-2)\tau(\theta_1)} \\ 0, & 0, & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0, & 0, & \cdots & 0 \end{bmatrix} \tag{10}$$

$$c_{ii} = c_s + \sigma^2, \qquad \text{for} \quad i = 1, 2, \cdots, M \tag{11}$$

where $c_{ik} \triangleq E\{y_i(n)y_k^*(n)\}$. By comparing this correlation c_{11} with the element r_{11} of $\boldsymbol{R}$ in (10), we have

$$\frac{|r_{11}|}{c_{11}} = 4\bar{p}(M-\bar{p})^{1/2}c_s\frac{\mathrm{SNR}}{\mathrm{SNR}+1} \tag{12}$$

where $\mathrm{SNR} \triangleq c_s/\sigma^2$. Thus for the case of $\mathrm{SNR} \geq -10$dB and $\bar{p} = \lfloor M/2 \rfloor$, as long as the signal power c_s is not less than $\bar{\alpha}_{c_s}$, we can find that $|r_{11}| \geq c_{11}$; whereas when there is no signal, we can obtain that $|r_{11}| = 0 < c_{11} = \sigma^2$, where $\bar{\alpha}_{c_s} \triangleq 11/4\bar{p}(M-\bar{p})^{1/2} \approx 5.5\sqrt{2}M^{-3/2}$, which is small due to the large M in many practice applications, and $\lfloor x \rfloor$ denotes the largest integer not greater than x.

3.3. Implementation of Proposed Method

When finite array data $\{\boldsymbol{y}(n)\}_{n=1}^N$ are available, the Hankel correlation matrices in (4) and (5) (and hence $\boldsymbol{\Phi}$) should be replaced with their estimates $\hat{\boldsymbol{\Phi}}_f$, $\hat{\bar{\boldsymbol{\Phi}}}_f$, $\hat{\boldsymbol{\Phi}}_b$, and $\hat{\bar{\boldsymbol{\Phi}}}_b$ (and hence $\hat{\boldsymbol{\Phi}}$), and then the sample estimate of auto-product $\hat{\boldsymbol{\Psi}}$ of $\hat{\boldsymbol{\Phi}}$ is given by

$$\hat{\boldsymbol{\Psi}} = \hat{\boldsymbol{\Phi}}\hat{\boldsymbol{\Phi}}^H = \hat{\boldsymbol{\Phi}}_f\hat{\boldsymbol{\Phi}}_f^H + \hat{\bar{\boldsymbol{\Phi}}}_f\hat{\bar{\boldsymbol{\Phi}}}_f^H + \hat{\boldsymbol{\Phi}}_b\hat{\boldsymbol{\Phi}}_b^H + \hat{\bar{\boldsymbol{\Phi}}}_b\hat{\bar{\boldsymbol{\Phi}}}_b^H. \quad (13)$$

As a result, when the number of snapshots N is not sufficiently large, the QR factor $\hat{\boldsymbol{R}}$ of $\hat{\boldsymbol{\Psi}}$ will be perturbed from its true value $\boldsymbol{R}$ in (7) and may become an upper-triangular and nonsingular matrix with full-rank due to the effect of estimation error. Hence the number of signals (i.e., the effective rank of $\hat{\boldsymbol{\Psi}}$) could not be determined simply by comparing the magnitude relation between the diagonal elements of $\hat{\boldsymbol{R}}$.

Now performing the QR decomposition with column pivoting to the matrix $\hat{\boldsymbol{\Psi}}$ in (13), we get

$$\hat{\boldsymbol{\Psi}}\boldsymbol{\Pi} = \hat{\boldsymbol{Q}}\hat{\boldsymbol{R}} = \hat{\boldsymbol{Q}}\begin{bmatrix} \hat{\boldsymbol{R}}_{11}, & \hat{\boldsymbol{R}}_{12} \\ \boldsymbol{O}_{(M-\bar{p}-p)\times p}, & \hat{\boldsymbol{R}}_{22} \end{bmatrix}\begin{matrix}\}p \\ \}M-\bar{p}-p\end{matrix} \quad (14)$$

where $\boldsymbol{\Pi}$ is an $(M-\bar{p})\times(M-\bar{p})$ permutation matrix, which is used to represent different methods of the QR decomposition with column interchanges (see Section 4.3 for the choice of $\boldsymbol{\Pi}$). Then by introducing an auxiliary quantity $\zeta(i)$ in terms of the non-zero elements of the ith row of QR factor $\hat{\boldsymbol{R}}$ as

$$\zeta(i) \triangleq \sum_{k=i}^{M-\bar{p}} |\hat{r}_{ik}| + \varepsilon, \quad \text{for } i = 1, 2, \cdots, M-\bar{p} \quad (15)$$

we can define a ratio criterion $\xi(i)$ as

$$\xi(i) \triangleq \frac{\zeta(i)}{\zeta(i+1)}, \quad \text{for } i = 1, 2, \cdots, M-\bar{p}-1 \quad (16)$$

where ε is an arbitrary and positive small constant (e.g., $\varepsilon = 10^{-10}$) for avoiding the possibly undetermined ratio of $0/0$ in (16). Thus the number of incident signals is determined as the value of the running index $i \in \{1, 2, \cdots, M-\bar{p}-1\}$ for which the criterion $\xi(i)$ is maximized, i.e.,

$$\hat{p} = \arg\max_i \xi(i). \quad (17)$$

Therefore the implementation of the proposed method and its computational complexity in MATLAB flops are summarized as:

1): Set the subarray size to $\bar{p} = \lfloor M/2 \rfloor$, which satisfies the condition that $p_{\max} \le \bar{p} = \lfloor M/2 \rfloor < M - p_{\max}$, where $p_{\max} = \lceil M/2 \rceil - 1$, and $\lceil x \rceil$ denotes the smallest integer not less than x. ....1 `flops`

2): Calculate the correlation vector $\hat{\boldsymbol{\varphi}}$ between $\boldsymbol{y}(n)$ and $y_M^*(n)$ and those of $\hat{\bar{\boldsymbol{\varphi}}}$ between $\boldsymbol{y}(n)$ and $y_1^*(n)$ as

$$\hat{\boldsymbol{\varphi}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{y}(n)y_M^*(n), \quad \hat{\bar{\boldsymbol{\varphi}}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{y}(n)y_1^*(n) \quad (18)$$

where $\hat{\boldsymbol{\varphi}} = [\hat{c}_{1M}, \hat{c}_{2M}, \cdots, \hat{c}_{MM}]^T$, and $\hat{\bar{\boldsymbol{\varphi}}} = [\hat{c}_{11}, \hat{c}_{21}, \cdots, \hat{c}_{M1}]^T$. ....$16NM$ `flops`

3): Form the estimated matrix $\hat{\boldsymbol{\Phi}}$ from $\hat{\boldsymbol{\varphi}}$ and $\hat{\bar{\boldsymbol{\varphi}}}$ as

$$\hat{\boldsymbol{\Phi}} = [\hat{\boldsymbol{\Phi}}_f, \hat{\bar{\boldsymbol{\Phi}}}_f, \hat{\boldsymbol{\Phi}}_b, \hat{\bar{\boldsymbol{\Phi}}}_b] \quad (19)$$

where $\hat{\boldsymbol{\Phi}}_f = \mathrm{Hank}\{\boldsymbol{h}_c, \boldsymbol{h}_r\}$, $\hat{\bar{\boldsymbol{\Phi}}}_f = \mathrm{Hank}\{\bar{\boldsymbol{h}}_c, \bar{\boldsymbol{h}}_r\}$, $\hat{\boldsymbol{\Phi}}_b = \boldsymbol{J}_{M-p}\hat{\bar{\boldsymbol{\Phi}}}_f^*\boldsymbol{J}_p$, $\hat{\bar{\boldsymbol{\Phi}}}_b = \boldsymbol{J}_{M-p}\hat{\boldsymbol{\Phi}}_f^*\boldsymbol{J}_p$, $\boldsymbol{h}_c = [\hat{c}_{1M}, \hat{c}_{2M}, \cdots, \hat{c}_{M-\bar{p},M}]^T$, $\boldsymbol{h}_r = [\hat{c}_{M-\bar{p},M}, \hat{c}_{M-\bar{p}+1,M}, \cdots, \hat{c}_{M-1,M}]^T$, $\bar{\boldsymbol{h}}_c = [\hat{c}_{21}, \hat{c}_{31}, \cdots, \hat{c}_{L1}]^T$, $\bar{\boldsymbol{h}}_r = [\hat{c}_{L1}, \hat{c}_{L+1,1}, \cdots, \hat{c}_{M1}]^T$, $\mathrm{Hank}\{\cdot\}$ denotes the Hankel operation, and $\boldsymbol{J}_m$ is an $m \times m$ counteridentity matrix.

4): Calculate the auto-product $\hat{\boldsymbol{\Psi}}$ of $\hat{\boldsymbol{\Phi}}$ as (13) and perform its QR decomposition with the permutation matrix $\boldsymbol{\Pi}$ as (14). ....$32\bar{p}(M-\bar{p})^2 + 14(M-\bar{p})^3 - 6(M-\bar{p})^2 + 22(M-\bar{p}) - 5$ `flops`

5): If $|\hat{r}_{11}| < \hat{c}_{11}$, the number of signal is estimated as $\hat{p} = 0$, and stop the detection procedure; otherwise $\hat{p} \ne 0$, and continue with the next step. ....2 `flops`

6): Calculate the ratio criterion $\xi(i)$ as (16) and determine the number of incident signals by using (17). ....$4.5(M-\bar{p})^2 + 3.5(M-\bar{p})^2 + M - \bar{p} - 1$ `flops`

4. STATISTICAL ANALYSIS

4.1. Consistency of Proposed Detection Criterion

An important characteristic of a detection scheme is its ability to provide an unbiased estimate of the number of incident signals for a large number of snapshots N. For investigating the consistency of the proposed MENSE, we firstly consider the asymptotical error of $\hat{\boldsymbol{\Psi}}$ in (13).

Theorem 1 *By dividing the total array into $\bar{p}+1$ overlapping "virtual" forward and backward subarrays with $M-\bar{p}$ sensors, the asymptotical error of $\hat{\boldsymbol{\Psi}}$ in (13) is given by*

$$E\{\hat{\boldsymbol{\Psi}} - \boldsymbol{\Psi}\} = \frac{1}{N}\left(c_{MM}(\boldsymbol{C}_1^f + \boldsymbol{C}_2^b) + c_{11}(\boldsymbol{C}_2^f + \boldsymbol{C}_1^b)\right) \quad (20)$$

where $\boldsymbol{C}_1^f$ and $\boldsymbol{C}_2^f$ are the spatially summed covariance matrices of the first and last $\bar{p}$ "virtual" forward subarrays, while $\boldsymbol{C}_1^b$ and $\boldsymbol{C}_2^b$ are those of the first and last $\bar{p}$ "virtual" backward subarrays.

Proof: Omitted (see [11] for details). ■

Thus as the number of snapshots N tends to infinity, the asymptotical error of the estimated auto-product $\hat{\boldsymbol{\Psi}}$ approaches zero, i.e., $\lim_{N\to\infty} E\{\hat{\boldsymbol{\Psi}} - \boldsymbol{\Psi}\} = \boldsymbol{O}_{(M-\bar{p})\times(M-\bar{p})}$. Then we can easily obtain the following theorem.

Theorem 2 *As the number of snapshots N tends toward infinity, the estimated number of incident signals obtained with (17) is consistent.*

Proof: Because the predetermined permutation matrix $\boldsymbol{\Pi}$ does not affect the statistical property of $\hat{\boldsymbol{\Psi}}\boldsymbol{\Pi}$ in (14) and the estimate $\hat{\boldsymbol{\Psi}}$ in (13) is asymptotically consistent, we easily find that the QR factor $\hat{\boldsymbol{R}}$ in (14) is asymptotically consistent, i.e., $\hat{\boldsymbol{R}} \xrightarrow{N\to\infty} \boldsymbol{R}$. Then from (7), (14), and (15), we can get $\zeta(i) \xrightarrow{N\to\infty} \bar{\zeta}_i + \varepsilon$ for $1 \le i \le p$ while $\zeta(i) \xrightarrow{N\to\infty} \varepsilon$ for $p < i \le M-\bar{p}$, where $\bar{\zeta}_i \triangleq \sum_{k=i}^{M-\bar{p}} |r_{ik}|$. Hence from (16), we have

$$\lim_{N\to\infty} \xi(i) = \begin{cases} \dfrac{\bar{\zeta}_i + \varepsilon}{\bar{\zeta}_{i+1} + \varepsilon} \approx \dfrac{\bar{\zeta}_i}{\bar{\zeta}_{i+1}} \triangleq \bar{c}_i, & \text{for } 1 \le i < p \\ \dfrac{\bar{\zeta}_p}{\varepsilon} + 1 \to \infty, & \text{for } i = p \\ \dfrac{\varepsilon}{\varepsilon} = 1, & \text{for } p < i \le M-\bar{p}-1 \end{cases} \quad (21)$$

where $\bar{c}_i$ is a positive constant. Thus it follows that $\xi(i) - \xi(p) < 0$ for $i < p$ and $i > p$ as $N \to \infty$; consequently the maximum is achieved at $i = p$. That is the probability of missing $P_m \triangleq \mathrm{Prob}(\hat{p} < p)$ and that of false alarm $P_{fa} \triangleq \mathrm{Prob}(\hat{p} > p)$ goes to zero asymptotically while the probability of correct detection approaches to one, when $N \to \infty$. Therefore we can conclude that the estimate $\hat{p}$ in (17) is asymptotically consistent, i.e., $\hat{p} = p$ with probability one (w.p.1) as $N \to \infty$. ■

4.2. Asymptotical Threshold for Detection

By defining an asymptotical auto-product $\boldsymbol{\Psi}_{as}$ of the estimate $\hat{\boldsymbol{\Psi}}$ in (13) as its expectation, from (20), we obtain

$$\boldsymbol{\Psi}_{as} = \boldsymbol{\Psi} + \frac{1}{N}\Big(c_{MM}(\bar{\boldsymbol{C}}_1^f + \bar{\boldsymbol{C}}_2^b) + c_{11}(\bar{\boldsymbol{C}}_2^f + \bar{\boldsymbol{C}}_1^b) + 2\bar{p}\sigma^2(c_{MM} + c_{11})\boldsymbol{I}_{M-\bar{p}}\Big) \quad (22)$$

where $c_{ii} = c_s\,|\bar{\rho}_i|^2 + \sigma^2$, while $\bar{\boldsymbol{C}}_1^f$, $\bar{\boldsymbol{C}}_2^f$, $\bar{\boldsymbol{C}}_1^b$, and $\bar{\boldsymbol{C}}_2^b$ are the four noise-free counterparts of $\boldsymbol{C}_1^f$, $\boldsymbol{C}_2^f$, $\boldsymbol{C}_1^b$, and $\boldsymbol{C}_2^b$ and their ranks equal the number of incident signals p. Then the rank of matrix $\boldsymbol{\Psi}_{as}$ in (22) is given by

$$\mathrm{rank}(\boldsymbol{\Psi}_{as}) = \begin{cases} p, & \text{for } \sigma^2 = 0 \text{ and/or } N \to \infty \\ M - \bar{p}, & \text{others} \end{cases}. \quad (23)$$

Hence the estimated number of signals is perturbed by the noise variance σ^2 and the number of snapshots N, and we can obtain a theoretical detection threshold for the SNR, N, or angular separation for correct detection by substituting the asymptotical matrix $\boldsymbol{\Psi}_{as}$ into the proposed algorithm.

Further when there is no incident signal, we can get

$$c_{11} = \sigma^2, \quad \text{and} \quad \boldsymbol{\Psi}_{as} = \frac{4}{N}\bar{p}\sigma^4\boldsymbol{I}_{M-\bar{p}}. \quad (24)$$

As a result, the element r_{11}^{as} of the QR upper-triangular factor $\boldsymbol{R}_{as}$ of $\boldsymbol{\Psi}_{as}$ in (24) is given by

$$r_{11}^{as} = \frac{4}{N}\bar{p}\sigma^4. \quad (25)$$

Then we can theoretically obtain the low threshold for the number of snapshots $\bar{N}$ for correctly detecting the absence of incident signal(s) as

$$\bar{N} > 4\bar{p}\sigma^4. \quad (26)$$

4.3. A Choice of QR Permutation Matrix

By reexpressing the matrix $\boldsymbol{\Psi}_{as}$ in its column vectors $\{\bar{\boldsymbol{\psi}}_i\}$, the linear independence between these columns can be got by considering the Gramian matrix $\boldsymbol{\Lambda}$ of $\boldsymbol{\Psi}_{as}$ given by

$$\boldsymbol{\Lambda} \triangleq \boldsymbol{\Psi}_{as}^H\boldsymbol{\Psi}_{as} = \{\lambda_{ik}\} \quad (27)$$

where $\lambda_{ik} = \bar{\boldsymbol{\psi}}_i^H\bar{\boldsymbol{\psi}}_k$, and $\boldsymbol{\Lambda}$ is centrosymmetric, Hermitian, and persymmetric as well as $\boldsymbol{\Psi}_{as}$. Then a quantitative measure of linear independence of columns $\{\bar{\boldsymbol{\psi}}_i\}$ is the dependency coefficient $\bar{\lambda}_{ik}$ defined as

$$\bar{\lambda}_{ik} \triangleq \frac{|\lambda_{ik}|}{\sqrt{\lambda_{ii}\lambda_{kk}}} \quad (28)$$

where $\bar{\lambda}_{ik} \le 1$. Thus by comparing the elements of a dependency coefficient matrix $\bar{\boldsymbol{\Lambda}} = \{\bar{\lambda}_{ik}\}$, we can form a permutation matrix $\boldsymbol{\Pi}$ to ensure the minimum linear dependency between the adjacent columns of $\boldsymbol{\Psi}_{as}\boldsymbol{\Pi}$. Further this predetermined permutation matrix $\boldsymbol{\Pi}$ can be used in (14) to improve the detection performance by lowing the detection threshold without increased computational complexity.

Remark B: The proposed determination of the permutation matrix may not be feasible in some applications, when the knowledge of asymptotical auto-produce $\boldsymbol{\Psi}_{as}$ is unavailable. Then an alternative is the column index maximum-difference bisection rule based scheme [8], which could possibly provide the shuffled columns with relatively small dependency by considering the special symmetries of $\boldsymbol{\Psi}$. □

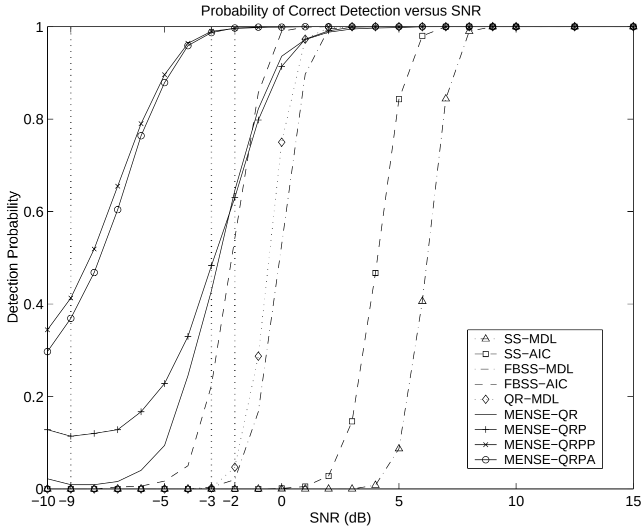

Figure 1. Probability of correct detection versus the SNR for Example 1 (vertical dotted line: detection threshold; $N = 64$, $M = 10$, and $p = 2$).

5. NUMERICAL EXAMPLES

The ULA with M sensors is separated by a half-wavelength, and the simulation results shown below are based on 1000 independent trials.

Example 1—Performance versus SNR: Two coherent signals with equal power ($c_s = 1$) arriving from $\theta_1 = 5°$ and $\theta_2 = 12°$, and their SNR is varied from -10 to 15dB. The number of sensors is $M = 10$, and the number of snapshots is $N = 64$. The subarray size is set at $m = 5$ for the SS- and FBSS-based methods.

From Section 4.3, we can obtain the predetermined permutation matrix as $\boldsymbol{\Pi} = [\boldsymbol{e}_1, \boldsymbol{e}_5, \boldsymbol{e}_2, \boldsymbol{e}_4, \boldsymbol{e}_3]$ (referred as the QRPA), where $\boldsymbol{e}_i$ is an $(M-\bar{p}) \times 1$ unit vector with a unity element at the ith location and zeros elsewhere. The QR-based method [8] is modified by combining the FBSS with the MDL criterion and referred as the QR-MDL method, where the QR decomposition with the predetermined permutation matrix based on the column index maximum-difference bisection rule [8] (referred as the QRPP) is also used. The probabilities of correct detection by the proposed algorithm with QRPA, QRPP, and QR (i.e., $\boldsymbol{\Pi} = \boldsymbol{I}_{M-\bar{p}}$) in terms of the SNR are shown in Fig. 1. The proposed algorithm with QR generally outperforms the SS- and FBSS-based methods [3]-[5] with EVD and the QR-MDL method. Moreover the performance of the proposed algorithm can be significantly improved at lower SNR by introducing the column pivoting into the QR decomposition (i.e., the QRPP, and QRPA) to further remedy the effect of additive noise, and the proposed QRPA has a similar effect as the QRPP [8]. Further the theoretically low threshold $\overline{\mathrm{SNR}}$ for correct detection is given by

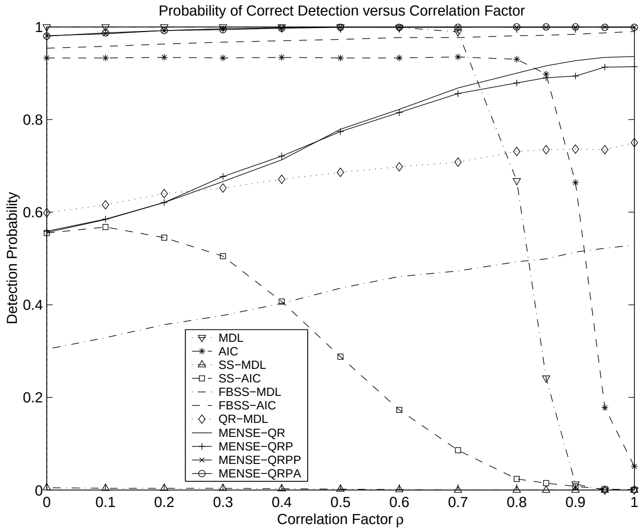

Figure 2. Probability of correct detection versus the correlation factor for Example 2 (SNR $= 0$dB, $N = 64$, $M = 10$, and $p = 2$).

-3 or -9dB, when the QR or QRPP/QRPA is used.

Example 2—Performance versus Correlation Factor: Then we evaluate the detection performance with respect to the correlation b etween the incident signals, where correlation factor ρ is varied from 0 to 1 (its phase is assumed to be zero), the SNR is set at 0dB, and the number of snapshots is fixed at $N = 64$, while the other parameters are the same as those in Example 1.

As shown in Fig. 2, although the proposed MENSE with QR or QRP is inferior to the FBSS-based AIC method, the MENSE algorithm with QRPP or QRPA is superior to the other methods in detecting strongly correlated signals, where the slight degradation for low correlations is due to the reduced dimension of the $(M-\bar{p})\times(M-\bar{p})$ sample matrix $\hat{\mathbf{\Psi}}$, where $M - \bar{p} = 5 < M$. Thus we can see that the proposed MENSE with QRPP/QRPA is insensitive to the correlation between incident signals.

Example 3—Detecting Absence of Incident Signal: There is no signal impinging on the array, i.e., $p = 0$, where $M = 10$, and the number of snapshots is varied from $N = 1$ to $N = 100$. The probability of false alarm with the proposed algorithm with QRPP/QR versus the number of snapshots is plotted in Fig. 3 for several noise variance $\sigma^2 = 1, 0.5, 0.3$, and 0.2. Generally the proposed algorithm with QRPP/QR has a rather smaller probability of false alarm and performs better than the EVD-based AIC and MDL methods for a smaller N when $\sigma^2 \leq 0.3$. Additionally as analyzed in (26), the theoretical threshold for $\bar{N}$ is given by $\bar{N} = 21, 11, 7$, and 5 for the noise variances $\sigma^2 = 1, 0.5, 0.3$, and 0.2, respectively.

6. CONCLUSION

A new QR-based method was proposed for estimating the number of narrowband signals impinging on a ULA, and its asymptotical consistency was studied. The proposed algorithm is superior in detecting closely-spaced signals with a small number of snapshots and/or at relatively low SNR. Moreover careful examinations revealed that the QRPP can be introduced into the proposed algorithm to significantly improve the detection performance without increased computational complexity and any *a priori* knowledge.

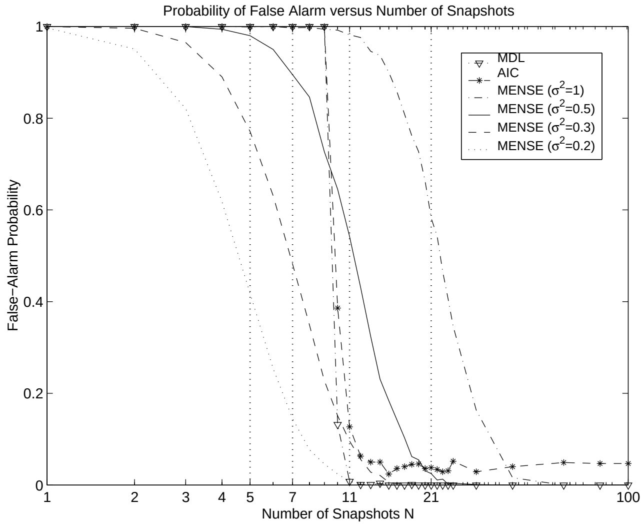

Figure 3. Probability of false alarm versus the number of snapshots for several noise variances for Example 3 (vertical dotted line: detection threshold; $M = 10$, and $p = 0$).

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