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RELATIONS BETWEEN GABOR TRANSFORMS AND FRACTIONAL FOURIER TRANSFORMS AND THEIR APPLICATIONS FOR SIGNAL PROCESSING

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ABSTRACT

Many wonderful relations between the Gabor transform and the fractional Fourier transform (FRFT), which is a generalization of the Fourier transform, are derived. First, we find that, as the Wigner distribution function (WDF), the FRFT is also equivalent to the rotation operation of the Gabor transform. We also derive the shifting, the projection, the power integration, and the energy sum relations between the Gabor transform and the FRFT. Since the Gabor transform is closely related to the FRFT, we can use it for analyzing the effect of the FRFT. Compared with the WDF, the Gabor transform does not have the problem of cross terms. It makes the Gabor transform a very powerful assistant tool for fractional sampling and designing the filter in the FRFT domain. Moreover, we show that any combination of the WDF and the Gabor transform also has the rotation relation with the FRFT.

1. INTRODUCTION

The fractional Fourier transform(FRFT) is defined as [1][2]:

$$O_F^{\alpha}\left(f(t)\right)=F_{\alpha}(u)$$
$$=\sqrt{\frac{1-j\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}e^{\frac{j}{2}u^2\cot\alpha-jut\csc\alpha+\frac{j}{2}t^2\cot\alpha}f(t)\,dt. \qquad (1)$$

It is a generalization of the Fourier transform (i.e., $\alpha=\pi/2$):

$$FT[f(t)]=O_F^{\pi/2}(f(t))=\sqrt{1/2\pi}\cdot\int_{-\infty}^{\infty}e^{-jut}f(t)\,dt. \qquad (2)$$

The FRFT can extend the utilities of the Fourier transform (FT). It is useful for filter design, pattern recognition, optics analysis, radar system analysis, communication, etc.

The Wigner distribution function (WDF) is [3]

$$W_f(t,\omega)=1/2\pi\cdot\int_{-\infty}^{\infty}f(t+\tau/2)\cdot f^*(t-\tau/2)\cdot e^{-j\omega\tau}\cdot d\tau. \ (3)$$

If $W_f(t,\omega)$ and $W_{F\alpha}(t,\omega)$ are the WDFs of $f(t)$ and its FRFT, $F_\alpha(u)$, then they have the following relation [4][5]:

$$W_{F_\alpha}(u,v)=W_f(u\cos\alpha-v\sin\alpha,\ u\sin\alpha+v\cos\alpha), \ (4)$$

That is, $W_{F\alpha}(u,v)$ is a rotation of $W_f(t,\omega)$. Since WDFs and the FRFTs have such a close relation, we often use WDFs to aid FRFTs for signal processing applications. For example, for filter design in the FRFT domain, WDFs are helpful for estimation the optimal order α [5]. Cohen's class distributions also have the rotation relation with the FRFT [11].

However, there is a problem for the WDF and Cohen's class distributions, i.e., "the cross term". This can be seen from Figs. 2(h), 3(b), and 5(h). It makes us hard to distinguish the signal part, the noise part, and the cross-term part. Thus it is hard to use the WDF and Cohen's class distributions to analyze the FRFT characters when the signal consists of multiple time-frequency components.

In this paper, we derive the relations between the Gabor transform and the FRFT. The Gabor transform is [6][7]:

$$G_f(t,\omega)=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{(\tau-t)^2}{2}}e^{-j\omega(\tau-\frac{t}{2})}f(\tau)\,d\tau. \qquad (5)$$

We find that, as the WDF, the FRFT also corresponds to the rotation operation for the Gabor transform. Thus as the WDF, the Gabor transform can also be used for analyzing the characters of signals in the FRFT domain. Comparing with the WDF, the Gabor transform has the advantages of

(1) avoiding the cross-term problem,
(2) less computation time.

Especially, due to the ability of avoiding the cross-term, the Gabor transform is more effective than the WDF for filter design (see Section 4). Without the misleading of the cross-term, the optimal parameter α of the FRFT will be much easier to determine. This problem perplexes the researchers on the field of the FRFT for many years. Now we can use the Gabor transform to solve it successfully.

Moreover, in Section 5, we find that, in addition to the WDF and the Gabor transform, if we do **arbitrary** combination for the Gabor transform and the WDF, the resultant transform also have the rotation relation with the FRFT.

2. ROTATION RELATION

[Rotation Relation]: If $F_\alpha(u)$ is the FRFT of $f(t)$, then their Gabor transforms (denoted by $G_f(t,\omega)$ and $G_{F\alpha}(u,v)$) have the relation of:

$$G_{F_\alpha}(u,v)=G_f(u\cos\alpha-v\sin\alpha,u\sin\alpha+v\cos\alpha). \quad (6)$$

That is, t*he FRFT corresponds to rotating the Gabor transform in the clockwise direction with angle α.*

(Proof): $G_{F_\alpha}(u,v)=\sqrt{1/2\pi}\int_{-\infty}^{\infty}e^{-\frac{(\tau-u)^2}{2}}e^{-jv(\tau-\frac{u}{2})}F_\alpha(\tau)d\tau$

$$=\frac{\sqrt{1-j\cot\alpha}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{(\tau-u)^2}{2}}e^{-jv(\tau-\frac{u}{2})}e^{\frac{j}{2}\tau^2\cot\alpha}e^{-j\tau x\csc\alpha}$$
$$e^{\frac{j}{2}x^2\cot\alpha}f(x)dx\,d\tau$$

$$= \frac{\sqrt{1-j\cot\alpha}}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{\tau^2\left(-\frac{1}{2}+\frac{j}{2}\cot\alpha\right)} e^{\tau(u-jv-jx\csc\alpha)}\,d\tau\right]\cdot e^{-\frac{u^2}{2}+\frac{juv}{2}+\frac{j}{2}x^2\cot\alpha} f(x)\,dx, \qquad (7)$$

Then, applying the fact that

$$\int_{-\infty}^{\infty} e^{-(a\tau^2+b\tau)}\,d\tau = \sqrt{\pi/a}\cdot e^{b^2/4a} \qquad (8)$$

(from Ref. [8]), we can rewrite (7) as:

$$G_{F_\alpha}(t,\omega)=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{\frac{(-u+jv+jx\csc\alpha)^2}{2(1-j\cot\alpha)}} e^{-\frac{u^2}{2}+\frac{juv}{2}+\frac{j}{2}x^2\cot\alpha} f(x)\,dx$$

$$=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{\frac{(u^2-v^2)\sin^2\alpha}{2}-vx\sin\alpha-\frac{x^2}{2}-jvt\sin^2\alpha-jux\sin\alpha}$$

$$e^{uv\sin\alpha\cos\alpha+ux\cos\alpha+\frac{j(u^2-v^2)\sin\alpha\cos\alpha}{2}-j\omega x\cos\alpha-\frac{jx^2\cot\alpha}{2}}$$

$$e^{-\frac{u^2}{2}+j\frac{uv}{2}+\frac{j}{2}x^2\cot\alpha} f(x)\,dx$$

$$=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-\frac{(u\cos\alpha-\omega\sin\alpha)^2}{2}+ux\cos\alpha-vx\sin\alpha-\frac{x^2}{2}}$$

$$e^{j\frac{uv(\cos^2-\sin^2)}{2}+\frac{j(u^2-v^2)\sin\alpha\cos\alpha}{2}-jx(u\sin\alpha+\omega\cos\alpha)}\,dx$$

$$=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{\frac{-[x-(u\cos\alpha-v\sin\alpha)]^2}{2}} \cdot e^{-j(u\sin\alpha+v\cos\alpha)\left(x-\frac{u\cos\alpha-v\sin\alpha}{2}\right)}\,dx \qquad (9)$$

It leads to (6) #

We perform an experiment in Fig. 1 to show that the FRFT is equivalent to the rotation relation of the Gabor transform.

Although the WDF also have the similar property, it has the problem of "cross term". If we use the Gabor transform instead of the WDF, **the cross term problem can be avoided**. That is, if

$$f(t) = s(t) + r(t) \qquad (10)$$

and $W_f(t, \omega)$, $W_s(t, \omega)$ and $W_r(t, \omega)$ are the WDFs of $f(t)$, $s(t)$, and $n(t)$, respectively, then

$$W_f(t, \omega) \neq W_s(t, \omega) + W_r(t, \omega). \qquad (11)$$

This is because the formula of the WDF contains the auto correlation term $f(t+\tau/2)f^*(t-\tau/2)$, see (2). If $f(t) = s(t) + r(t)$,

$$f(t+\tau/2)f^*(t-\tau/2) = s(t+\tau/2)s^*(t-\tau/2) + r(t+\tau/2)r^*(t-\tau/2) + s(t+\tau/2)r^*(t-\tau/2) + r(t+\tau/2)s^*(t-\tau/2).$$

Due to the cross terms $s(t+\tau/2)r^*(t-\tau/2)$ and $r(t+\tau/2)s^*(t-\tau/2)$, $W_f(t, \omega)$ will not be the sum of $W_s(t, \omega)$ and $W_r(t, \omega)$.

In contrast, when using the Gabor transform, the auto correlation is avoided, (see (5)). If $G_f(t, \omega)$, $G_s(t, \omega)$ and $G_r(t, \omega)$ are the Gabor transforms of $f(t)$, $s(t)$, and $n(t)$, then

$$G_f(t, \omega) = G_s(t, \omega) + G_r(t, \omega), \qquad (12)$$

and the problem of cross term can be avoided. In Fig. 2, we do some experiments to show the Gabor transforms and the WDFs of $s(t)$, $r(t)$, and $f(t) = s(t)+r(t)$, where

$$s(t)=\exp\left(jt^2/10-j3t\right) \text{ for } -9 \le t \le 1,$$
$$s(t) = 0 \text{ otherwise}, \qquad (13)$$

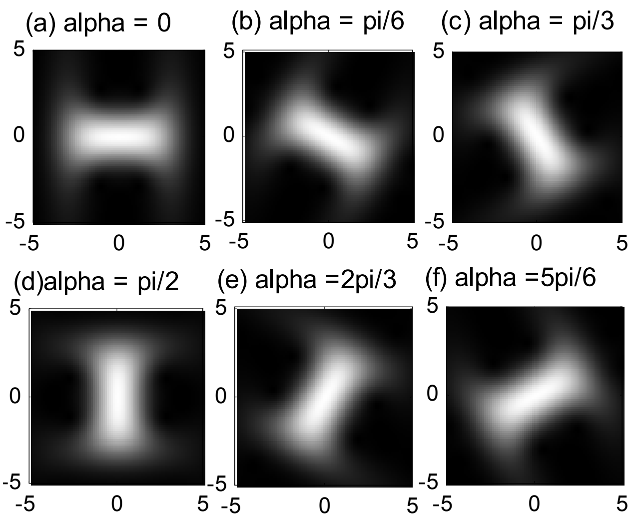

Fig. 1 The Gabor transform of $F_\alpha(u)$ where $F_\alpha(u)$ is the FRFT of a rectangular function $f(t)$, $f(t) = 1$ for $|t| \le 3$, and $f(t) = 0$ otherwise. We use gray level to show the magnitude of $G_{F\alpha}(u, v)$.

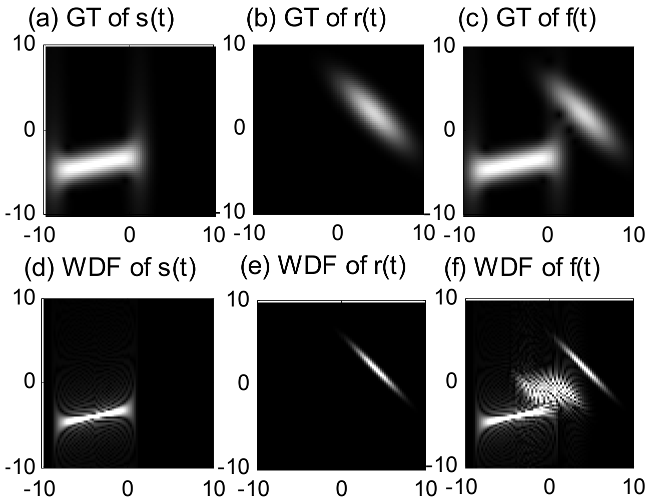

Fig. 2 The Gabor transform (GT) and the Wigner distribution functions (WDF) of $s(t)$, $r(t)$, and $f(t) = s(t)+r(t)$. Note the WDF has the "cross term problem" but the GT does not.

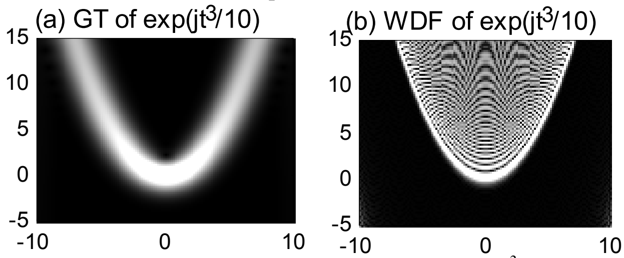

Fig. 3 The GT and the WDF for $f(t) = \exp(jt^3/10)$.

$$r(t)=\exp\left(-jt^2/2+j6t\right)\exp\left[-(t-4)^2/10\right]. \qquad (14)$$

Although when $f(t) = \exp(jat^2+jbt)$, the resolution of the Gabor transform may not be as good as the WDT, it has an important advantage of avoiding the cross-term.

Moreover, if

$$f(t) = \exp(jat^k + \text{remained terms}), \quad k \ge 3 \text{ and } a \neq 0, \qquad (15)$$

the Gabor transform will have better resolution than the WDF, as the example in Fig. 3.

3. PROPERTIES AND IMPLEMENTATION

[Recovery Relation]: We can recover $f(t)$ from $G_f(t, \omega)$ by:

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} G_f(t,\omega)e^{\frac{j\omega t}{2}}\,d\omega = f(t). \quad (16)$$

This relation can be generalized into the case of the FRFT. In (16), if we replace $f(t)$ by $F_\alpha(u)$, then

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} G_{F_\alpha}(u,v)e^{\frac{juv}{2}}\,dv = F_\alpha(u),$$

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} G_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha)e^{\frac{juv}{2}}\,dv = F_\alpha(u).$$

(applying (6)). (17)

That is, we can obtain $F_\alpha(u)$ from the Gabor transform of $f(t)$ if we do the scaled inverse FT along the direction of

$$(-\sin\alpha, \cos\alpha). \quad (18)$$

[Projection Relation]: The recovery relation for $f(t)$ in (16) can be generalized into:

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} G_f(t,\omega)e^{jk\omega t}\,d\omega = e^{-\frac{(k-1/2)^2t^2}{2}} f((1/2+k)t). \quad (19)$$

In (19), if we replace $f(t)$ by $F_\alpha(u)$ and apply (6), it becomes

$$\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} G_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha)e^{jkuv}\,dv = e^{-\frac{(k-1/2)^2t^2}{2}} F_\alpha((k+1/2)u) \quad (20)$$

When $k = 1/2$, it becomes (17).

[Power Integration Relation]: If we integrate the power of $G_f(t, \omega)$ along ω-axis, then

$$\int_{-\infty}^{\infty} |G_f(t,\omega)|^2 d\omega = \int_{-\infty}^{\infty} e^{-(\tau-t)^2} |f(\tau)|^2 d\tau. \quad (21)$$

For the case of the FRFT, we replace $f(t)$ in (22) by $F_\alpha(u)$:

$$\int_{-\infty}^{\infty} |G_{F_\alpha}(u,v)|^2 dv = \int_{-\infty}^{\infty} e^{-(\tau-u)^2} |F_\alpha(\tau)|^2 d\tau, \quad (22)$$

Then we apply (6),

$$\int_{-\infty}^{\infty} |G_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha)|^2 dv = \int_{-\infty}^{\infty} e^{-(\tau-u)^2} |F_\alpha(\tau)|^2 d\tau. \quad (23)$$

Therefore, the power integrating along the line of

$$u(\cos\alpha, \sin\alpha) + v(-\sin\alpha, \cos\alpha), \quad v \in (-\infty, \infty), \quad (24)$$

will be the **local energy** of $F_\alpha(\tau)$ around $\tau = u$.

[Energy Sum Relation]: From (23),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G_f(t,\omega)|^2 d\omega\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\tau-t)^2} |f(\tau)|^2 d\tau\,dt,$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G_f(t,\omega)|^2 d\omega\,dt = \sqrt{\pi}\int_{-\infty}^{\infty} |f(\tau)|^2 d\tau. \quad (25)$$

Then, from the fact that the energy is preserved after rotation and the Parseval's theory of the FRFT [1], (25) can be generalized as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G_{F_\beta}(t,\omega)|^2 d\omega\,dt = \sqrt{\pi}\int_{-\infty}^{\infty} |F_\alpha(u)|^2 du \quad (26)$$

for any α, β.

[Power Decayed Relation]: If $F_\alpha(u) = 0$ for $u > u_0$, then

$$\int |G_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha)|^2 dv < e^{-(u-u_0)^2} \int |G_f(u_0\cos\alpha - v\sin\alpha, u_0\sin\alpha + v\cos\alpha)|^2 dv \quad (27)$$

[Shifting and Modulation Relations]: If

$$h(t) = f(t - t_0),$$

then their Gabor transforms have the following relation:

$$G_h(t,\omega) = G_f(t - t_0, \omega)e^{-j\omega t_0/2}, \quad (28)$$

If $h(t) = f(t)\exp(j\omega_0 t)$, then

$$G_h(t,\omega) = G_f(t, \omega - \omega_0)e^{j\omega_0 t/2}, \quad (29)$$

Generally, if $H_\alpha(u)$ and $F_\alpha(u)$ are FRFTs of $h(t)$ and $f(t)$ and

$$H_\alpha(u) = F_\alpha(u - u_0),$$

then

$$G_h(t,\omega) = G_f(t - u_0\cos\alpha, \omega - u_0\sin\alpha)e^{\frac{j(t\sin\alpha - \omega\cos\alpha)u_0}{2}}. \quad (30)$$

[Discretization] When doing digital implementation, in (5), we set $\tau = m\delta_t$, $t = n\delta_t$, and $\omega = s\delta_\omega$. To preserve the rotation property and other properties, it is proper to choose

$\delta_t = \delta_\omega < 2\pi / \mathrm{Max}(B_\alpha)$,

where B_α is the supporting width of $|F_\alpha(u)| > \Delta$. (31)

Moreover, since $e^{-x^2/2} < 0.0001$ when $|x| > 4.292$, from (5), to save the computational time, instead of varying m from $-\infty$ to ∞, we can set the range of m as

$$n - 4.292/\delta_t < m < n + 4.292/\delta_t. \quad (32)$$

4. APPLICATIONS FOR SIGNAL PROCESSING IN THE FRFT DOMAIN

Since the Gabor transform is closely related to the FRFT, we can use it as an assistant tool for signal processing in the FRFT domain. We give two examples: (A) fractional sampling and (B) fractional filter design to show how to use the Gabor transform together with FRFTs for signal processing.

When using the FRFT to do signal sampling [9], we first try to find α such that the supporting of $|F_\alpha(u)|$ is minimal

supporting Ω_α: $F_\alpha(u) < \Delta$ if $u \notin \Omega_\alpha$,

optimal $\alpha \rightarrow$ width(Ω_α) is minimal. (33)

When $f(t)$ has only one time-frequency (T-F) component, we can use the WDF to estimate width(Ω_α) and hence search the optimal α. However, when $f(t)$ has two or more T-F components, as the example in Fig. 2(f), using the WDF to estimate α may not be proper. In this case, since the "orient" for each of the T-F component is different, it is proper to separate $f(t)$ into several T-F components and determine the optimal α for each of the component. Due to the cross term, this work is hard to be done by the WDF. Note that, in Fig. 2(f), it is hard to conclude whether the central region around (0, 0) is "the third T-F component" of $f(t)$ or just the cross term of the left part and the right part.

In contrast, when using the Gabor transform, since there is no cross term, we can easily conclude how may T-F components $f(t)$ contains and decompose $f(t)$ into the summation of these components. For the example in Fig. 2,

(a) First, we decompose $G_f(t, \omega)$ in Fig. 2(c) into the left part and the right part, which corresponds to $s(t)$ and $r(t)$.

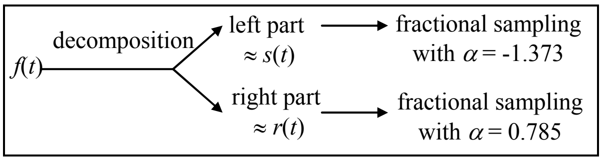

Fig. 4 Using the FRFT together with the time-frequency component decomposition by the Gabor transform to sample the signal $f(t)$ (see Fig. 2) in the FRFT domain.

(b) Then we use the power integration relation to estimate the optimal order α (defined by (33)) for the left and the right T-F components

$$\alpha_{\text{left}} = -1.373, \qquad \alpha_{\text{right}} = 0.785. \qquad (34)$$

(c) Then, we apply the algorithm in [9] to do fractional sampling for each of the components. When the number of sampling points is fixed to 10, the reconstruction errors are

- Using the conventional sampling theory: $err = 2.763\%$
- Sampling $f(t)$ in the FRFT domain:, $err = 1.306\%$,
- Sampling $f(t)$ in the FRFT domain with the aid of the Gabor transform: $err = 0.061\%$. (35)

Then we discuss how to use the Gabor transform to design the filter in the FRFT domain. It is known that we can use the FRFT instead of the FT for filter design [9], i.e.,

$$r(t) = O_F^{-\alpha}\left\{O_F^{\alpha}[h(t)]T_\alpha(u)\right\} \qquad (36)$$

where $h(t)$ and $r(t)$ are the input and the output of the filter, respectively. Although the way for searching the optimal transfer function $T_\alpha(u)$ has been developed [1][10], it lacks an efficient way to determine the optimal order α. It seems that the WDF is helpful for determining α and the cutoff line, however, due to the cross term problem, using the WDF is not suitable for the case where $h(t)$ consists of a lot of time-frequency (T-F) components.

In this paper, we find that the Gabor transform also has the rotation relation with the FRFT and it can avoid the cross term problem. This hints that we can use the Gabor transform instead of the WDF for fractional filter design.

We give an example in Fig. 5. The input signal is

$$s(t) = 2\cos(5t)\exp(-t^2/10) \text{ (shown in Fig. 5(a)).} \qquad (37)$$

It is interfered by the following noise

$$n(t) = 0.5e^{j0.23t^2} + e^{j0.3t^2}\cos(10t), \qquad (38)$$

and $h(t) = s(t) + n(t)$ is plotted in Fig. 5(b). We want to recover $s(t)$ from $h(t)$ by the filter in the FRFT domain.

The WDFs of $s(t)$ and $h(t)$ are plotted in Fig. 5(g)(h). In Fig. 5(h), the signal parts, the noise parts, and the cross term parts are mixed together. It is hard to know how to separate the signal parts from the noise parts after observing Fig. 5(h).

In contrast, when doing the Gabor transform, since the problem of cross term can be avoided, in Fig. 5(d), the signal parts and the noise parts are separated clearly. We can use the following four lines (in Fig. 5(e)) to separate them:

L1: $-2(\cos\alpha, \sin\alpha) + k(-\sin\alpha, \cos\alpha)$, $\alpha = -1.14$,
L2: $2(\cos\alpha, \sin\alpha) + k(-\sin\alpha, \cos\alpha)$
L3: $-7.5(\cos\beta, \sin\beta) + k(-\sin\beta, \cos\beta)$, $\beta = -1.03$,
L4: $7.5(\cos\beta, \sin\beta) + k(-\sin\beta, \cos\beta)$, (39)

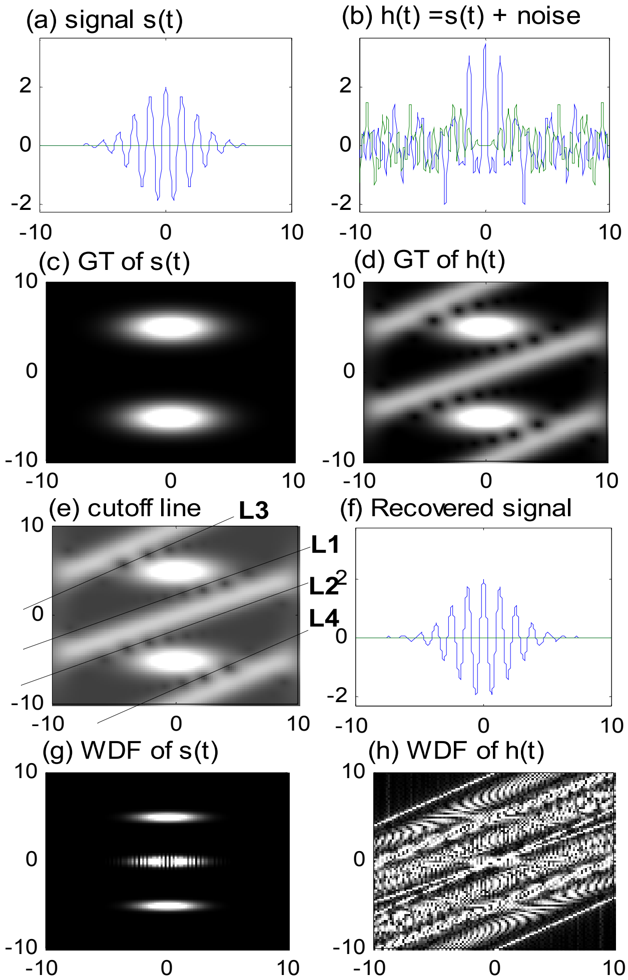

Fig. 5 Using the FRFT with the Gabor transform for filter design.

Therefore, we can use the following process to filter out the noise and recover $s(t)$ from $h(t)$. Step (b) comes from L1 and L2 and Step (d) comes from L3 and L4:

(a) $H_\alpha(u) = O_F^{\alpha}[h(t)]$, (40)

(b) $H_{1,\alpha}(u) = H_\alpha(u)$ for $|u| \geq 2$, $H_{1,\alpha}(u) = 0$ for $|u| < 2$, (41)

(c) $H_{1,\beta}(u) = O_F^{\beta-\alpha}[H_{1,\alpha}(t)]$, (42)

(d) $H_{2,\beta}(u) = H_{1,\beta}(u)$ for $|u| \leq 7.5$,
$H_{2,\beta}(u) = 0$ for $|u| > 7.5$, (43)

(e) $s(t) \approx O_F^{-\beta}[H_{2,\beta}(u)]$. (44)

The recovered signal is plotted in Fig. 5(f). It is very close to the original signal $s(t)$ and the error is only 0.049%.

We give another example in Fig. 6. The input is in Fig. 6(a). It is interfered by the noise with the 3rd order phase.

$$n(t) = 0.9\exp(j0.025t^3 - j5t). \qquad (45)$$

When using the WDF, the noise and the signal parts are mixed, see Fig. 6(c). When we use the Gabor transform, the noise and the signal parts are obviously separable, see Fig 6(d). Since the noise and the signal parts can be separated by three lines, we can use three fractional filters to filter out the noise. The orders of FRFTs for these fractional filters can be determined by the slopes of the cutoff lines (α = 0.6947, 1.5708, –0.6947). With them, we can recover the original signal with a very small error, see Fig. 6(b).

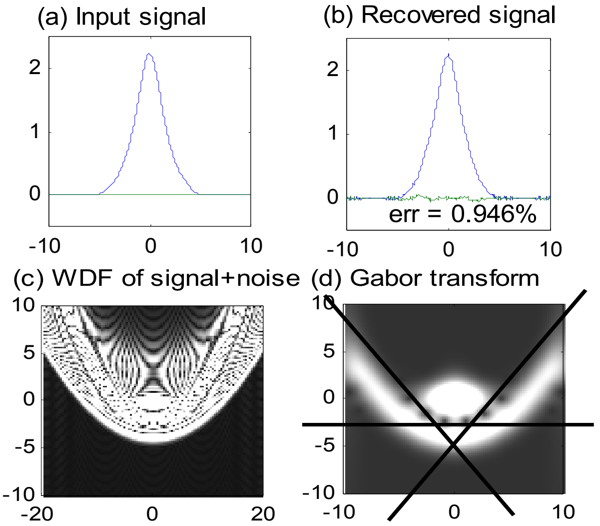

Fig. 6 Using the FRFT filter to filter output the 3rd order phase exponential function

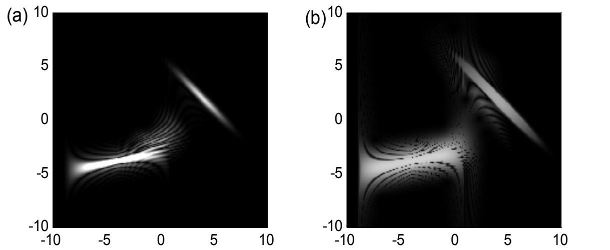

Fig. 7 The GWTs of $f(t)$ where $f(t) = s(t)+r(t)$ is defined in (13) and (14) and the GWTs are defined in (49) and (50).

5. GABOR-WIGNER TRANSFORM

We have known that both the Gabor transform and the WDF has the rotation relation with the FRFT. In fact, if we combine the Gabor transform with the WDF properly, the new time-frequency distribution also has the rotation relation with the FRFT.

[Combination Theorem]: Suppose that $p(x, y)$ is **any** function with two variables. If we define a new time frequency $C_f(t, \omega)$ (We call it the **Gabor-Wigner transform** (**GWT**)) that has the following relation with the Gabor transform $G_f(t, \omega)$ and the WDF $W_f(t, \omega)$:

$$C_f(t,\omega) = p\left(G_f(t,\omega), W_f(t,\omega)\right), \quad (46)$$

then $C_f(t, \omega)$ also has the rotation relation with the FRFT:

$$C_{F_\alpha}(u,v) = C_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha), \quad (47)$$

where $C_f(t, \omega)$ and $C_{F\alpha}(u, v)$ are the GWT of $f(t)$ and its FRFT, $F_\alpha(u)$, respectively.

[Proof]: From (46),

$$C_{F_\alpha}(u,v) = p\left(G_{F_\alpha}(u,v), W_{F_\alpha}(u,v)\right) \quad (48)$$

$$= p\big(G_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha),$$
$$W_f(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha)\big)$$
$$= C_{F_\alpha}(u\cos\alpha - v\sin\alpha, u\sin\alpha + v\cos\alpha). \quad \#$$

Since the GWT also has rotation relation, thus it is possible to use it instead of the WDF and the Gabor transform for signal analysis in the FRFT domain. In fact, if $p(x, y)$ is chosen properly, the resultant GWT will **combine the advantages of the Gabor transform and the WDF**. In Fig. 7, we perform several experiments. For Fig. 7(a),

$$C_f(t,\omega) = G_f(t,\omega) W_f(t,\omega), \quad (49)$$

For Fig. 7(b),

$$C_f(t,\omega) = \min\left(|G_f(t,\omega)|^2, |W_f(t,\omega)|\right), \quad (50)$$

That is, $p(x, y) = xy$ in (49) and $p(x, y) = \min(|x|^2, |y|)$ in (50).

Compare Fig. 7 with Fig. 2, we find that, as the Gabor transform, when using the GWT, the cross term problem can also be avoided. Moreover, its resolution is obviously better than that of the Gabor transform. It combines both the advantage of the WDF (higher resolution) and the advantage of the Gabor transform (no cross term).

6. CONCLUSIONS

We derived several interesting relations between the FRFT and the Gabor transform, including the rotation relation, the recovery relation, and the power integration relation. Since the Gabor transform can avoid the cross term, which is a serious problem for the WDF, we could use it instead of the WDF to do signal process in the FRFT domain, such as fractional sampling and the fractional filter design.

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