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Design of Interpolation Functions Using Iterative Methods

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ABSTRACT

We propose a new method to compensate for the distortion of any interpolation function. This is a hybrid method based on the iterative method proposed by one of the authors where modular harmonics are utilized instead of simple lowpass filter. The hybrid method is also improved by the Chebyshev acceleration algorithm. The proposed technique drastically improves the convergence rate with less computational complexity while it is robust to additive noise. This method could be used in any 1-D signals which must be interpolated during the process.

1. INTRODUCTION

There are many applications in digital signal processing and communication systems that require the reconstruction of an analog signal from its discrete time samples using D/A converters. Several methods with different names were introduced in the literature during 1970's and 1980's (refer to [1] for a complete survey of interpolation techniques). Sample-and-Hold (S&H: zero-order-hold) and Linear Interpolation (LI: first-order-hold) were dominant methods before. Polynomial interpolation and B-splines (Cubic splines) are the usual interpolation functions [2-4].

These interpolators create some distortion at the Nyquist rate after low pass filtering, especially when S&H or LI are utilized. The advantage of these types of interpolators is their simplicity making them proper for practical use in iterative techniques. There are several methods to compensate for this kind of distortion such as inverse sinc filtering, over-sampling, nonlinear and adaptive algorithms [5-6], a modular method [7], and successive approximation using iterative methods [8-9]. The modular method is compared to the inverse sinc filtering in [7]. This reference shows that using a few numbers of modules, the performance of the modular method excels the inverse filtering as far as noise is concerned. Over-sampling is not a practical solution due to its bandwidth requirement. The iterative method [8] outperforms the modular method at the cost of more computation.

We are proposing a hybrid method that combines the iterative and the modular methods. The advantages of this hybrid method are fast convergence rate, low complexity and delay, and robustness against additive noise. In the sequel, we will briefly describe the modular and the iterative methods and then prove the convergence of the hybrid method. Noise analysis and sensitivity will be discussed, and simulation results and comparison with other methods will be presented. Computational complexity comparison will conclude this paper.

2. BACKGROUND

Before explaining our proposed method, we need to go over the previous methods. Below we will describe the modular and the iterative methods before the proposed hybrid method.

2.1. The Modular Method

In this method [7], the interpolated signal *s(t)* (S&H, LI or Spline) is mixed with a sum of cosine waves and then passed through a lowpass filter. The output can be written as

$$\hat{x}_s(t) = s(t)[1 + 2\cos(2\pi t/T) + \ldots 2\cos(2N\pi t/T)] \quad (1)$$

$$\hat{x}(t) \ = \ \mathrm{LPF}\{\hat{x}_s(t)\} \quad (2)$$

where T is the sampling period and N represents the number of modules which are multiplied to *s(t)*. Moreover, *s(t)* is the interpolation of *x(t)* samples and $\hat{x}_s(t)$ is the approximation of ideal *x(t)* samples. In fact, the sum

$$1 + 2\sum_{s=1}^{N}\cos(\frac{2\pi st}{T}) = \frac{\sin\pi(N+1)t/T}{\sin\pi t/T} \quad (3)$$

is the first $N+1$ harmonics of the Fourier series expansion of the impulse train. Therefore, as N increases, $\hat{x}_s(t)$ converges to the ideal samples of *x(t)* and $\hat{x}(t)$ converges to *x(t)*. Fig. 1 describes the modular method.

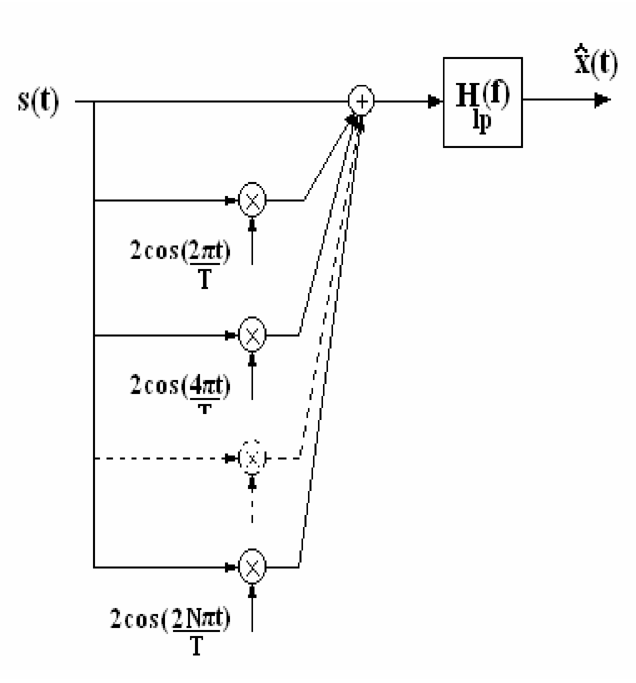

Figure 1: The Modular Method

2.2. Iterative Method

The iterative method is given by

$$x_{k+1}(t)=\lambda \mathrm{PS}x(t)+(I-\lambda \mathrm{PS})x_k(t) \tag{4}$$

where λ is the relaxation parameter that determines the convergence rate. $x_k(t)$ is the kth iteration and $x_0(t)$ can be any function of time. But $x_0(t)=\hat{x}(t)=\mathrm{PS}x(t)$ is the natural choice for faster convergence. P is a band-limiting operator and S is a sampling process, e.g., S can be S&H or LI and P can be a lowpass filter. Redefining the operators G=λPS and E=I-G, we can rewrite (4) as

$$x_{k+1}=\mathrm{G}x+(\mathrm{I}-\mathrm{G})x_k=\hat{x}(t)+\mathrm{E}x_k \tag{5}$$

It is straightforward to show that (5) can be written as

$$x_k(t)=(\mathrm{E}^k+\mathrm{E}^{k-1}+\ldots+\mathrm{E}+\mathrm{I})\hat{x}(t) \tag{6}$$

If G is a linear operator, we can write

$$\mathrm{E}^k+\mathrm{E}^{k-1}+\ldots+\mathrm{E}+\mathrm{I}=\frac{\mathrm{I}-\mathrm{E}^{k+1}}{\mathrm{I}-\mathrm{E}} \tag{7}$$

If the operator norm ||E||<1, by increasing the number of iterations (k), (7) approaches the inverse system G^{-1} [10], therefore, $x_k(t)$ will converge to $x(t)$. By setting λ correctly, we can satisfy this constraint in general. A detailed proof of convergence and its relationship with λ is given in [8-10].

3. THE PROPOSED HYBRID METHOD

3.1. Description

We propose to combine the modular and iterative Methods. We incorporate the modular Method in each iteration step before applying the lowpass filter. In other words, the P operator is no longer a simple lowpass filter. Instead, it consists of a Modular interpolator. Because the Modular method outperforms simple lowpass filtering, it makes the iterative method to converge much faster. We shall see in the simulation section that only one module is sufficient for this phenomenal improvement. Below we shall prove the convergence for the S&H interpolation. The proof for other types of interpolation is similar.

3.1.1. Proof of Convergence for the S&H interpolation:

For the P and S operators defined for S&H, we can write

$$x_{k+1}(t)=\lambda \mathrm{P}x_m(t)+\mathrm{P}x_k(t)-\lambda \mathrm{P}x_{mk}(t) \tag{8}$$

where x_m and x_{mk} are defined as

$$x_m(t)=[1+2\sum_{s=1}^{N}\cos(\frac{2\pi st}{T})]\sum_n x(nT)\Pi(\frac{t-n}{T}-1/2) \tag{9}$$

$$x_{mk}(t)=[1+2\sum_{s=1}^{N}\cos(\frac{2\pi st}{T})]\sum_n x_k(nT)\Pi(\frac{t-n}{T}-1/2) \tag{10}$$

$\Pi(.)$ is a rectangular function used for S&H. T is the sampling interval, and N is the number of mixed harmonics. $x_k(t)$ will converge to $x(t)$ in the limit, if the operator E is a contraction, i.e., ||E||<1. This implies [8]

$$||x_{k+1}-x_k||<||x_k-x_{k-1}|| \tag{11}$$

Substituting (8) in (11), we can write

$$||x_k-x_{k-1}-\lambda \mathrm{PS}(x_k-x_{k-1})||\le r\,||x_k-x_{k-1}|| \tag{12}$$

where $0\le r<1$. The left-hand side of (12) can be written in the frequency domain, using the Parseval's theorem:

$$||X_k(f)-X_{k-1}(f)-\lambda\Pi(fT)\sum_{s=-N}^{N}\mathrm{sinc}(fT-s).\sum_i X_k(f-\frac{i}{T})-X_{k-1}(f-\frac{i}{T})|| \tag{13}$$

where $\Pi(fT)$ is an ideal lowpass filter with the cut-off frequency of $f=1/(2T)$. By assuming the sampling rate is at the Nyquist rate, (13) becomes

$$||[X_k(f)-X_{k-1}(f)].\{1-\lambda\sum_{s=-N}^{N}\mathrm{sinc}(fT-s)\}|| \tag{14}$$

Hence,

$$||X_k(f)-X_{k-1}(f)||\le \max|1-\lambda\sum_{s=-N}^{N}\mathrm{sinc}(fT-s)| \tag{15}$$

To satisfy (12), it is required that

$$0 < r = \max|1 - \lambda \sum_{s=-N}^{N} \mathrm{sinc}(fT\text{-}s)| < 1 \qquad (16)$$

In the case of one module utilization, the maximum occurs at $f_c=1/2T$ and at this point for $\lambda=1$, we get $r = 0.06 < 1$. Thus, the proposed hybrid method converges to the original signal. At $\lambda=1$ comparing $r=0.06$ with r=1-sinc(1/2)=0.36 in the conventional iterative method, we expect a drastic convergence rate improvement. By increasing the number of harmonics in the modular technique r tends to zero for $\lambda=1$, and thus faster convergence is expected.

For the best convergence rate, the relaxation parameter λ should be chosen to minimize r, thus at the Nyquist rate, the optimal value for λ for the case of one additional harmonic is

$$\lambda_{opt} = 1/(\sum_{s=-1}^{1} \mathrm{sinc}(\tfrac{1}{2}\text{-}s)) \cong 0.94 \qquad (17)$$

For other types of interpolations, the derivations are similar.

3.2. Noise Analysis

Suppose that the proposed hybrid method is implemented and used in a noisy environment. For the sake of analysis white noise is added to the signal at each iteration stage. In this section we will analyze and compare the effect of noise for the hybrid and traditional methods.

For the traditional iterative method, we can write

$$x_{k+1}(t) = \lambda \mathrm{P} x_s(t) + \mathrm{P} x_k(\mathrm{t}) - \lambda \mathrm{P} x_{sk}(t) + n_k(t) \qquad (18)$$

where $n_k(t)$ is the additive white Gaussian noise in the *kth* iteration, $x_s(t)$ and $x_{sk}(t)$ are the S&H versions of $x(t)$ and $x_k(t)$, respectively. The necessary constraint on the convergence is the contraction inequality given in (11). Substituting (18) in (11), we obtain

$$\| x_k - x_{k-1} - \lambda PS(x_k - x_{k-1}) + n_k - n_{k-1} \| \le r \| x_k - x_{k-1} \| \qquad (19)$$

By invoking the triangle inequality, it is sufficient to have

$$\| x_k - x_{k-1} - \lambda PS(x_k - x_{k-1}) \| + \| n_k - n_{k-1} \| \le r \| x_k - x_{k-1} \| \qquad (20)$$

Assuming $n_k(t)$ and $n_{k-1}(t)$ are uncorrelated, similar to the previous section, the following inequality

$$\| X_k - X_{k-1} \| (r - |1 - \lambda \mathrm{sinc}(fT)|_{\max}) \ge 2 \| n \| \qquad (22)$$

should be satisfied for $0 \le r < 1$. Considering the optimum λ, for the worst case we have

$$\| n \| \le \frac{1}{2} \| X_k - X_{k-1} \| (\lambda \sin c(1/2)) \cong 0.382 \| X_k - X_{k-1} \| \qquad (23)$$

This implies that as long as the noise standard deviation satisfies (23), the iteration will converge.

Now, consider the suggested method. Similar to (22) we can state that

$$\| X_k - X_{k-1} \| (r - |1 - \lambda \sum_{s=-N}^{N} \mathrm{sinc}(fT\text{-}s)|_{\max}) \ge 2 \| n \| \qquad (24)$$

is a sufficient constraint for the convergence. In the case of one module implementation and optimized relaxation factor, for the worst case we have

$$\|n\| \le \tfrac{1}{2} \|X_k - X_{k-1}\| \qquad (25)$$

Comparing (25) to (23), we conclude that the proposed hybrid method can tolerate more noise power. Notice that the iterative method always tends to the ideally reconstructed baseband signal. Therefore, the iterative method cannot eliminate additive noise in the baseband.

3.3. Acceleration of the iterative method

One of disadvantages of the traditional iterative method is its low convergence rate, even for the optimum relaxation factor. The iterative and thus the hybrid method can be accelerated by utilizing the two previous iterations. Chebyshev acceleration method [11] is given by

$$x_n = (x_1 + x_{n-1} - \tfrac{2}{A+B} PSx_{n-1} - x_{n-2})\lambda_n + x_{n-2} \qquad n>1, \qquad (26)$$

where $x_0 = \hat{x}$ and $x_1=2/(A+B)x_0$. P and S are the operators as defined for the iterative method. The constants A and B are frame bounds [11], and should be selected properly for acceptable performance. There is no unique optimum pair of A and B, so they have to be selected by experimental methods, but for one time before running the system.

The parameter λ_n can be calculated as follows

$$\lambda_n = (1 - \tfrac{\rho^2}{4} \lambda_{n-1})^{-1} \qquad (27)$$

Where ρ is defined as

$$\rho = \frac{B - A}{B + A} \qquad (28)$$

A detailed proof of the convergence based on the Chebyshev polynomials is presented in [11]. The extension of the proof for the hybrid method is straightforward.

The acceleration method improves the iterative method with little additional complexity. Notice that the parameter λ_n only depends on the constants A and B and it is sufficient to calculate the λ vector once and save in memory for next uses.

4. SIMULATION RESULTS AND DISCUSSION

To have a fair comparison, initial bandlimited signals are produced randomly (uniform distribution), and the performance of each method is averaged over 50 signals. Signal length is set to 4096 and there are 64 samples representing the Nyquist rate for the discrete signal. The initial signals are lowpass filtered version of pseudo-random signals. FFT/IFFT block size is set to 128. To show the significance of this method, the sampling process is performed at the Nyquist rate. The performance criterion for our simulations is the Signal to Noise Ratio (SNR) in dB.

SNR is calculated for interior points and 10% of the end points are ignored, so that the transient errors are avoided. In all cases the optimized relaxation factor is utilized. The interpolator type used in the iterations is zero-order hold, because of its simplicity. Other types of interpolators such as spline can be utilized, but they require heavy calculations.

As illustrated in Fig. 2, the SNR increases monotonically in dB as the number of iterations increases. For the classical iterative method, after two iterations, the SNR of about 40 dB is achieved. This means 23dB improvement with respect to simple filtering of sample-and-hold signal. After such number of iterations, the hybrid method achieves a performance of about 84dB in terms of SNR. On the other hand, the accelerated hybrid method after two iterations reaches 97dB and 100dB for one module and two modules, respectively. Hence, the accelerated hybrid method improvement is about 83dB for only one harmonic, this is quite impressive in real engineering applications. In comparison with the Modular method, one step of the conventional iterative method has approximately the same performance with the modular method. But the performance improvement of one step of the hybrid method is nearly twice the modular method of two modules. In comparison with the state-of-the-art methods such as fast B-spline, the fast B-spline attains the performance of about 38dB with an interpolator of order 15, but the accelerated hybrid method reaches to the SNR of 60dB after a single iteration.

4.1. Noisy Environment

To study the effect of noise on the convergence rate and maximum achievable SNR, assume that a white Gaussian noise is added to each iteration step. This is a model of the electronic devices that generate thermal noise. Furthermore, a uniformly distributed noise is added to the signal at each iteration step as the quantization noise.

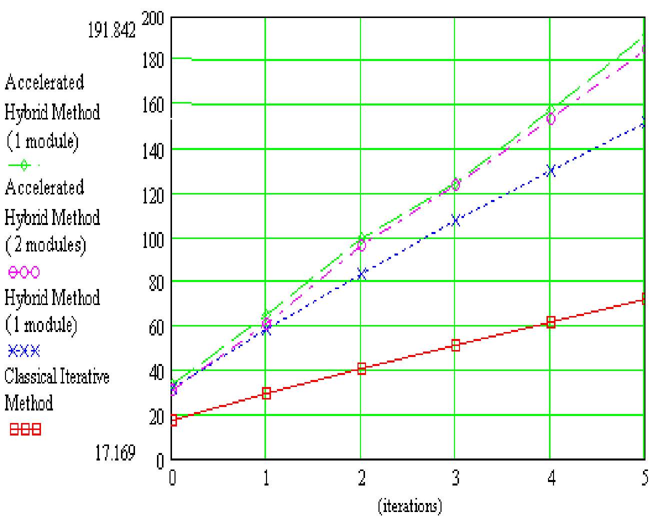

Figure 2: SNR vs. the number of iterations for different methods

Fig. 3 shows that after a few iterations, the SNR plot will reach its maximum value. After this maximum point, the SNR gets worse due to error propagation. Despite the degradations, Fig. 3 shows that the hybrid method is still more robust than the conventional method.

4.2. Computational Complexity

The major advantage of the proposed hybrid method is its higher rate of convergence with less overall computational complexity. The conventional iterative method requires $M(4Log_2N+2)$ real additions and $M(2Log_2N+1)$ real multiplications per sample. Where M is the number of iterations and N is the FFT block size. But the accelerated hybrid method with one module requires $M(4Log_2N+4)$ additions and $M(2Log_2N+3)$ multiplications per sample.

Although the number of computations for the hybrid case in each iterative step is more, but because with fewer numbers of iterations it achieves the same results, its overall computational load is considerably less.

In comparison with the fast B-spline interpolator [3], the B-spline interpolator requires $3n+1$ real additions and $2[n/2]+1$ real multiplications per sample value. Where n is the spline order. As mentioned before, in the case of B-spline of order 15, an output SNR of about 38dB is achieved while the proposed method obtain the performance of about 60dB in terms of SNR after just one iteration. The proposed method needs 32 additions and 17 multiplications per sample, but the fast B-spline method requires 46 additions and 15 multiplications. Thus, the proposed method drastically outperforms the fast B-spline method by only a little additional complexity.

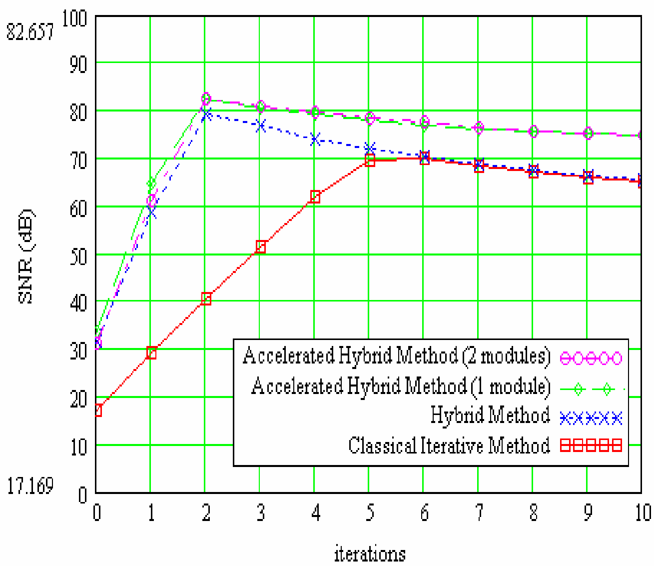

Figure 3: SNR vs. number of iterations for the different proposed methods in the presence of noise

5. CONCLUSION

We proposed a hybrid technique based on the iterative and the modular methods to compensate for the interpolation distortion such as S&H and LI. We have proven theoretically that this method converges much faster than the conventional methods. Furthermore, the robustness of this technique in noisy environment is shown to be better. The hybrid method can be improved by Chebyshev acceleration method. Simulation results confirm the theoretical findings.

The proposed improved hybrid technique has improved the convergence rate significantly for S&H interpolation. An improvement of about 60dB in terms of SNR is achieved for the same number of iterations. The hybrid method is also more favorable in terms of computational complexity with respect to the previous iterative methods, and outperforms state-of-the-art interpolation methods such as B-spline.

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