MODELING ULTRASOUND IMAGES WITH THE GENERALIZED K MODEL.

Torbjørn Eltoft

Department of Physics, Faculty of Science, University of Tromsø, 9037 Tromsø.
& Norut IT, P.O.box 6434, Forskningsparken, 9294 Tromsø, Norway.
E-mail:torbjorn.eltoft@phys.uit.no

ABSTRACT

In this paper we interpret the statistics of ultrasonic backscatter in the framework of a normal variance-mean mixture model. This is done by considering the complex envelope of the echo signal as a double stochastic circular Gaussian variable, in which both the variance and the mean are linearly scaled by a stochastic factor Z. By assuming Z to be Γ distributed, we re-derive the generalized K distribution, and present a new iterative algorithm for estimating its parameters. We also derive a maximum a posteriori (MAP) filter based on the generalized K model. The appropriateness of the generalized K model in representing the local amplitude statistics of medical ultrasound images, and the filtering performance of the the new MAP filter, are tested in some preliminary experiments.

1. INTRODUCTION

Many statistical models have been proposed to model the amplitude statistics of ultrasound signals. Both *Rayleigh* and *non-Rayleigh* models have been considered. The Rayleigh model is associated with the Gaussian or diffuse scattering model, due to the fact that the complex envelope of the signal in this case is circularly Gaussian distributed [1]. The most well-known non-Rayleigh amplitude model is the K distribution [2]. This distribution has been proposed to model echoes from tissue containing scatterers with variable concentration and non-uniform cross section [3]. Among other non-Rayleigh models we mention the Nakagami distribution [4], the homodyned and the generalized K-distributions [5], and the recently introduced Rician Inverse Gaussian (RiIG) distribution [6, 7].

In this paper we revisit the generalized K model. We show that by formulating the statistics of ultrasonic scattering as a normal variance-mean mixture model, in which both the variance and the mean are linearly scaled by a stochastic factor Z, we can easily calculate the pdf of the generalized K distribution. Furthermore, this approach also allows us to obtain a new iterative algorithm for estimating the parameters, and to derive new maximum a posteriori speckle filter.

The paper is organized as follows. In the next section, we present the normal variance-mean mixture model and derive the probability density function of the generalized K distribution. In section 3, we present a new algorithm for estimating the model parameters from data. In section 4, we apply the algorithm to fit the generalized K model to the local amplitude statistics of real medical ultrasound images. We derive a MAP filter based on this model, and show its filtering performance. In section 5 we give some conclusions.

2. THE GENERALIZED K DISTRIBUTION

2.1 Normal variance-mean mixture models

In [8] it was shown that if the probability density function (pdf) of some random variable Y, $p_Y(y)$, is symmetric about zero, and the derivatives of $p_Y(y)$ satisfy

$$\left(-\frac{d}{dy}\right)^k p_Y(y) \geq 0 \quad \text{for } y > 0, \tag{1}$$

then there exist independent variables X and Z, with X being a standard normal variable, such that

$$Y = \sqrt{Z}X. \tag{2}$$

The variable Z is allowed to take on only positive values. A random variable Y, which can be expressed as in (2), is referred to as *a normal variance mixture model*, or *a scale mixture of Gaussians*. If the mean of Y is non-zero, (2) may be modified by adding a scalar *mu* corresponding to the actual mean value. The marginal pdf of Y is obtained by integrating the conditional distribution $p_{Y|Z}(y|Z)$ over $p_Z(z)$, as in (3) below:

$$\begin{aligned} p_Y(y) &= \int_0^\infty p_{Y|Z}(y|Z=z)p_Z(z)\,dz \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{(y-m)^2}{2z}\right) p_Z(z)\,dz. \end{aligned} \tag{3}$$

A more general model, known as a normal variance-mean mixture model, was introduced in [9]. A 1-D normal variance-mean mixture variable is in its most general form expressed as

$$Y = m + bZ + \sqrt{Z}X, \tag{4}$$

where X and Z are defined as above, and b is a scalar parameter. Hence, in this model, both the mean and the variance of Y are varying linearly according to the stochastic variable Z.

The multidimensional extension of the generative model described above, is straight forward. Let $\mathbf{X}$ be a d-dimensional, zero mean Gaussian variable with covariance matrix equal to the identity matrix. Let furthermore, $\mathbf{\Gamma} \in \mathscr{R}^{d\times d}$ be a positive definite matrix with determinant $\det \mathbf{\Gamma} = 1$, and let Z be a scalar random variable with pdf $p_Z(z)$, which can attain only positive values. We now generate a new variable $\mathbf{Y}$ as *a multivariate variance-mean mixture variate* according to

$$\mathbf{Y} = \mathbf{m} + \mathbf{b}Z + \sqrt{Z}\mathbf{\Gamma}^{\frac{1}{2}}\mathbf{X}, \tag{5}$$

where $\mathbf{m}$ is a location vector, $\mathbf{b}$ is a vector parameter accounting for the linear scaling of the mean of $\mathbf{Y}$ as function of Z. The matrix $\mathbf{\Gamma}$ defines the internal covariance structure of the variables of $\mathbf{Y}$. For this reason we will refer to this matrix as the covariance structure matrix. To obtain the marginal pdf of $\mathbf{Y}$, we have to perform an integration similar to the one in (3) over the prior distribution $p_Z(z)$. The integral which must be computed is accordingly given as

$$p_{\mathbf{Y}}(\mathbf{y}) = \int_0^\infty \frac{1}{(2\pi z)^{\frac{d}{2}}} \times \exp\left(-\frac{1}{2z}(\mathbf{y}-\mathbf{m}-\mathbf{b}z)^t \mathbf{\Gamma}^{-1}(\mathbf{y}-\mathbf{m}-\mathbf{b}z)\right) p_Z(z)dz. \quad (6)$$

In general, probability density functions generated according to (5) will turn out as so-called sparse distributions, i.e., they are peaked at their mode, and have heavier tails than the Gaussian pdf.

2.2 Deriving the generalized K pdf

In ultrasound, a complex backscattered signal Y is often represented in terms of its quadrature components, i.e.

$$Y = Y_1 + jY_2 = Re^{j\Phi}, \quad (7)$$

where ϕ is the phase, R is the amplitude, and Y_1 and Y_2 are referred to as the *in-phase* (I) and *quadrature* (Q) components, respectively. Let us consider Y as a 2-D normal variance-mean mixture signal, i.e. $\mathbf{Y} = [Y_1, Y_2]^t$, where $\mathbf{Y}$ is generated as in (5). The signals Y_1 and Y_2 are usually assumed to be uncorrelated, and with equal power. Accordingly, the covariance structure matrix Γ should be given as the 2-D identity matrix, i.e.

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (8)$$

Note that, for a given value of the scale variable Z, the Y_1- and Y_2-components are independent Gaussian variables with non-zero means and variances equal to z. However, when Z is itself a random variable, the unconditional distribution of Y_1 and Y_2 is non-Gaussian, and the variables are statistically dependent.

We now continue to derive the distribution for the amplitude R in the case when $\mathbf{m} = 0$ in (5), but $\mathbf{b}$ is non-zero. The covariance structure matrix is still the 2-D identity matrix, and the pdf of the Γ distribution is written as

$$p_Z(z) = \left(\frac{\alpha}{\mu_Z}\right)^\alpha \frac{z^{\alpha-1}}{\Gamma(\alpha)} \exp\left(-\frac{\alpha z}{\mu_Z}\right). \quad (9)$$

Then we have

$$\mathbf{Y} = \mathbf{b}Z + \sqrt{Z}\mathbf{X}, \quad (10)$$

and the simultaneous pdf of $(y_1, y_2|Z)$ becomes

$$p_{Y_1,Y_2|Z}(y_1,y_2|Z) = \frac{1}{2\pi z} \exp\left(-\frac{(y_1-b_1z)^2+(y_2-b_2z)^2}{2z}\right), \quad (11)$$

where $[b_1, b_2]^t = [b\cos(\omega), b\sin(\omega)]^t$. ω is the angle of $\mathbf{b}$ with respect to the Y_1-axis, and b is the norm of $\mathbf{b}$.

The envelope of $\mathbf{Y}$ is now defined as

$$R = \sqrt{(bz\cos(\omega)+Y_1)^2 + (bz\sin(\omega)+Y_1)^2}, \quad (12)$$

and the corresponding angle variable is

$$\Phi = \tan^{-1}\left(\frac{bz\sin(\omega)+Y_1}{bz\cos(\omega)+Y_2)}\right). \quad (13)$$

We now switch to polar coordinates, and integrate over Φ. The resulting Z conditioned pdf for R becomes

$$p_{R|Z}(r|Z) = \frac{r}{z}\exp\left(-\frac{r^2+b^2z^2}{2z}\right) I_0(br), \quad (14)$$

where $I_0(\cdot)$ is the modified Bessel function of first kind. This is the well-known Rice distribution. The marginal distribution for R is obtained by integrating over the prior distribution for Z. Choosing Z to be Γ distributed as in (9), we get

$$p_R(r) = \left(\frac{\alpha}{\mu_Z}\right)^\alpha \frac{r}{\Gamma(\alpha)} I_0(br) \times \int_0^\infty z^{\alpha-2} \exp\left(-\frac{1}{2}\left(\frac{r^2}{z} + (b^2 + \frac{2\alpha}{\mu_Z})z\right)\right) dz. \quad (15)$$

This integral can be solved in closed form, and the resulting pdf is given as

$$p_R(r) = \frac{2}{\Gamma(\alpha)}\left(\frac{\alpha r}{\mu_Z}\right)^\alpha \frac{K_{\alpha-1}\left(r\sqrt{b^2+\frac{2\alpha}{\mu_Z}}\right)}{\left(\sqrt{b^2+\frac{2\alpha}{\mu_Z}}\right)^{\alpha-1}} I_0(rb), \quad (16)$$

where $K_\nu(\cdot)$ is the modified Bessel function of second kind, and order ν. This is a normalized, valid probability density function, defined by the three parameters α, μ_Z, and b. It is known in the literature as the generalized K distribution. In the next subsection we will describe a procedure for estimating these parameters from data.

3. PARAMETER ESTIMATION

The generalized K distribution is defined through a latent stochastic variable Z. We will below show how its parameters may be estimated in an iterative maximum likelihood approach, using an EM type algorithm. In this case, the E-step involves updating the first and second order moments of $Z|R$, and the M-step updates the parameters.

The latent variable Z is in our case Γ distributed, and using the expression in (9) for the pdf of Z, its k-th order moments are

$$E\{Z^k\} = \left(\frac{\mu_Z}{\alpha}\right)^k \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. \quad (17)$$

From (17) we immediately see that μ_Z and α are obtained as

$$\mu_Z = E\{Z\}, \quad (18)$$

and

$$\alpha = \frac{1}{E\{Z^2\}E\{Z\}^2 - 1}. \quad (19)$$

Furthermore, using Bayes rule we find that the posterior distribution $Z|R$ has a pdf given as

$$p_{Z|R}(z|R) = \frac{p_{R|Z}(r|Z)\,p_Z(z)}{p_R(r)} = \left(\frac{\sqrt{b^2+\frac{2\alpha}{\mu_Z}}}{r}\right)^{\alpha-1}$$

$$\times \frac{z^{(\alpha-1)-1}}{2K_\alpha(r\sqrt{b^2+\frac{2\alpha}{\mu_Z}})} \exp\left(-\frac{1}{2}(\frac{r^2}{z}+(b^2+\frac{2\alpha}{\mu_Z})z)\right). \quad (20)$$

The expression for $p_{Z|R}(z|R)$ is recognized as a Generalized Inverse Gaussian pdf with parameters $\{\alpha-1, r, \sqrt{b^2+\frac{2\alpha}{\mu_Z}}\}$, i.e $Z|R \sim \text{GIG}(z;\alpha-1,r,\sqrt{2+\frac{2\alpha}{\mu_Z}})$. The kth-order moments of a $\text{GIG}(u;\alpha,\delta,\gamma))$ distribution is [9]

$$E\{U^k\} = \left(\frac{\delta}{\gamma}\right)^k \frac{K_{\alpha+k}(\delta\gamma)}{K_\alpha(\delta\gamma)}. \quad (21)$$

We furthermore note that the second order moment of the Rice pdf in (14) is $2Z+b^2Z^2$, which when averaged over a Γ-distributed Z gives

$$E\{R^2\} = 2\mu_Z + b^2\frac{\mu_Z^2(\alpha+1)}{\alpha}. \quad (22)$$

For given values of α and μ_Z, b may be estimated from $E\{R^2\}$ as

$$\hat{b} = \sqrt{\frac{\alpha}{\alpha+1}\frac{E\{R^2\}-2\mu_Z}{\mu_Z^2}}. \quad (23)$$

From (21) we have that, for a given observation r_i,

$$\eta_i = E\{Z|r_i\} = \frac{r_i}{\sqrt{b^2+\frac{2\alpha}{\mu_Z}}}\frac{K_\alpha(r_i\sqrt{b^2+\frac{2\alpha}{\mu_Z}})}{K_{\alpha-1}(r_i\sqrt{b^2+\frac{2\alpha}{\mu_Z}})}, \quad (24)$$

and

$$\xi_i = E\{Z^2|r_i\} = \left(\frac{r_i}{\sqrt{b^2+\frac{2\alpha}{\mu_Z}}}\right)^2 \frac{K_{\alpha+1}(r_i\sqrt{b^2+\frac{2\alpha}{\mu_Z}})}{K_{\alpha-1}(r_i\sqrt{b^2+\frac{2\alpha}{\mu_Z}})}. \quad (25)$$

Given N observations, we define

$$\bar{\eta} = \frac{1}{N}\sum_{i=1}^{N}\eta_i, \quad (26)$$

and

$$\bar{\xi} = \frac{1}{N}\sum_{i=1}^{N}\xi_i. \quad (27)$$

Regarding $\bar{\eta}$ and $\bar{\xi}$ as estimates for $E\{Z\}$ and $E\{Z^2\}$, respectively, estimates for μ_Z and α may be obtained from (18) and (19).

Iterative parameter estimation procedure
(i): Calculate $E\{R^2\} = \frac{1}{N}\sum_{i=1}^{N} r_i^2$.
(ii): Set $l=0$. Select some initial estimates for the parameters $\hat{\alpha}_l$ and $\hat{\mu_Z}_l$.
(iii): Set l=l+1. Estimate $\hat{b_l}$ using (23).
(iv): Calculate η_i and ξ_i using equations (24) and (25).
(v): Calculate $\bar{\eta}$ and $\bar{\xi}$ (26) and (27).
(vi): Estimate $\hat{\mu_Z}_l$ and $\hat{\alpha}_l$ using (18) and (19).
(vii): Repeat steps *(iii)*- *(vii)* until convergence.

A convergence test may be defined based on the changes in the parameters at each iteration stage, or by looking at the change in the log-likelihood score. Our experience is that convergence is achieved very fast, often in less than 10 iterations.

Remark: We observe that the probability model, which is known as the K distribution, is a special case of the generalized K distribution with $b=0$. Hence, all the derivations given above are valid for the K distribution. In fact, the iterative estimation procedure should be considered as a useful alternative to existing estimators for the parameters of the K model.

4. EXPERIMENTAL ANALYSIS

In the experimental analysis, we examine two aspects of the generalized K model. First, we examine the appropriateness of the generalized K model to represent the amplitude statistics of linearly scaled medical ultrasound data. This is done by visually comparing the model pdf to locally generated histograms. We use the original K model as a reference. We also perform a cross-validation log-likelihood test to compare the goodness of fit of these to models. We then construct a maximum a posteriori (MAP) speckle filter based on the generalized K model, and show that the preliminary results reveal excellent filtering performance. Homogeneous regions of the image are smoothed, whereas details seems to be preserved.

4.1 Modeling the amplitude statistics of local image segments

We select regions of size 50 × 50 pixels from some ultrasound images covering a human kidney, a human liver, and a human heart. Using the iterative algorithm described in the previous section, we estimate the parameters of the generalized K model, and plot the resulting pdf on top of the actual local histogram. The same process is repeated for the K model, using the modified algorithm, i.e. requiring $b=0$. Four examples are displayed in Fig.1. These examples first of all show that both algorithms converge to pdfs, which have shapes close to the shapes of the histograms. Secondly, the plots in Fig.1 show that the generalized K model has better overall fit to the histograms than the K model. This is to be expected, since this is the more general model, which has the K as a special case. The histograms of the four examples have some diversity in shape, and as can be seen, the generalized K model is in fact able to quite accurately fit to all shapes. It is also interesting to note that the parameter estimation algorithm of the generalized K model converges in most cases in less than 10 iterations, whereas the algorithm for the K model uses a lot more.
In the log-likelihood tests we randomly picked 100 image segments of size 40 × 40, and calculated the associated log-likelihood values using a 10-fold cross-validation approach. In 70 % of the cases, the generalized K model was selected as the best model. In those cases where the K model won, the log-likelihood values of the two models were very similar.

4.2 Speckle filtering

Speckle filtering in ultrasound images is usually classified into two types; compounding techniques and filtering [10]. In the compounding techniques, a series of images of one object are sampled at different times, with different ultrasound

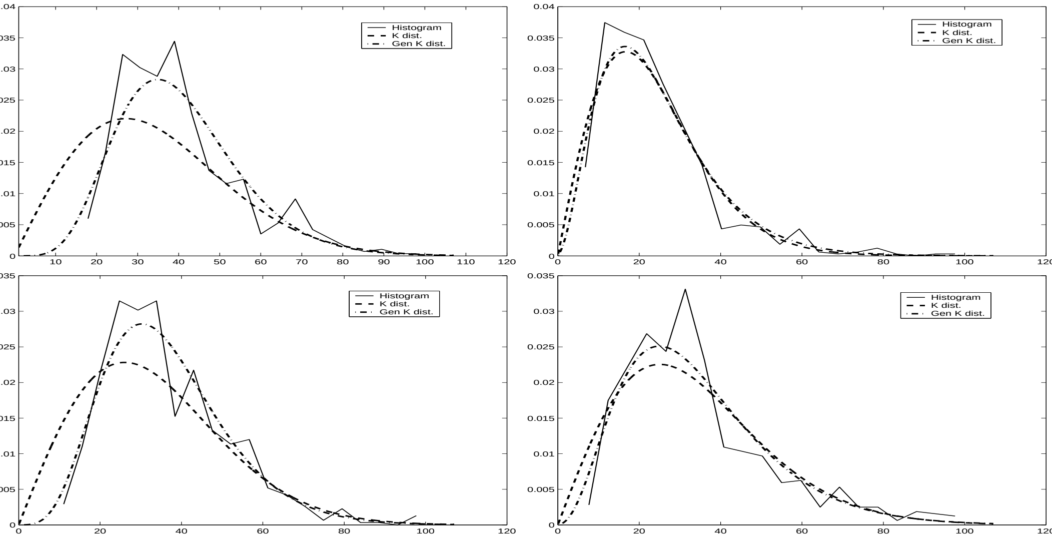

Figure 1: The plots show the results of fitting the generalized K (dashed-dotted) and the K (dashed) distributions to some local amplitude histograms (solid) of some medical ultrasound image segments. The images cover a human kidney (upper left), human liver (upper right), and human heart (both lower panels).

frequencies, or different scan directions, and subsequently merged to form a composite image. This process is known to reduce the spatial resolution of the image. In the filtering techniques a moving filter kernel is used to reduce the speckle noise. Examples of some well-known such filters are the Wiener filter, the median filter, the adaptive median filter [11], and the minimum mean squared error filters [12, 13].

Some speckle filters have been developed using a Bayesian approach. In the Bayesian filtering approach the image is filtered by performing a statistical estimation of the speckle-free image based on a statistical model for the image formation process. The estimated speckle-free amplitude of each pixel, $\hat{z}[x,y]$, where $[x,y]$ is the spatial location, is obtained as the most likely amplitude value, given an observed value $r[x,y]$. This procedure is known as maximum a posteriori filtering, because the solution corresponds to the location mode in the posterior distribution. Neglecting the spatial coordinates, the estimation is stated in mathematical terms as

$$\hat{z} = \arg\max_{z} \; p_{Z|R}(z|r). \qquad (28)$$

$p_{Z|R}(z|r)$ is generally not known, and one has to restate the expression in terms of known pdfs using Bayes rule. When the generalized K distribution is used to model the speckled data in the image domain, the $p_{Z|R}(z|r)$ of (20) is used in (28), and the solution is found by differentiating (20) (or the logarithm of (20)) with respect to z, and finding the zero corresponding to a positive z value. This gives

$$\hat{z} = \frac{(\alpha-2) + \sqrt{(\alpha-2)^2 + r^2(b^2 + \frac{2\alpha}{\mu_Z})}}{(b^2 + \frac{2\alpha}{\mu_Z})}. \qquad (29)$$

Hence, in a local window around the pixel at location $[x,y]$, the generalized K parameters are estimated from the observed data. The parameters of the corresponding posterior pdf are thus also given, and the estimated speckle-free amplitude is found as the location of the peak of $p_{Z|R}(z|r)$, with μ_Z and α replaced by their local estimates, and r replaced by the observed $r[x,y]$. The amplitude estimate is then found from (29).

In the literature a MAP filter based on the K model, is generally referred to as the Γ-MAP filter, because of the assumption of a Γ distributed Z [14]. This filter has been widely used in speckle filtering of ultrasound and SAR images. In the case of the K model the Z variable is physically related to the normalized backscatter cross section. In the generalized K model, the Z variable is, as can be seen from (10), also effecting the mean of the complex backscattered signal, and the interpretation is not so obvious. We will refer to the MAP filter presented in (29) as *the generalized Γ-MAP filter*.

Fig.2 shows the result of applying the generalized Γ-MAP filter to a medical ultrasound image of the human heart. The images are from left to right: the original image, the result of filtering with a window size of 7×7, 9×9, and 15×15, respectively. We observe that the filter process has smoothed homogeneous areas, but in regions where there are abrupt changes and details, this information seems to have been preserved. We also see that a larger window size will result in more smoothing, as would be expected.

5. CONCLUSION

We have shown that the generalized K distribution can be derived in the framework of a normal variance-mean mixture model, with a Γ distributed scale factor. In the context of ultrasound scattering from biological tissues, this model corresponds to viewing the scattering process as continuous Brownian motion with drift, as opposed to the usual discrete random walk model. In the paper we have shown that our approach readily gives us an iterative procedure for estimating the parameters from data, and we are also able to obtain a closed form maximum a posteriori speckle filter based on the generalized K model. Some initial tests shows that the

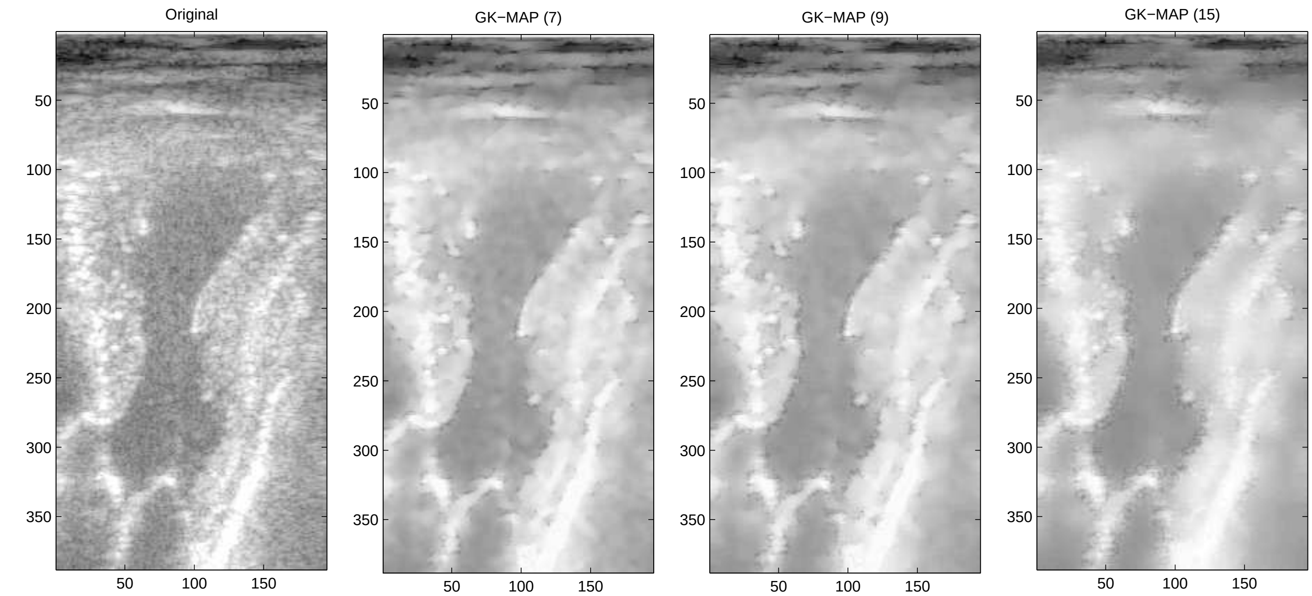

Figure 2: The result of filtering an ultrasound image of the human heart. The images are from left to right: Original image, filtered image with window size 7×7, 9×9, and 15×15, respectively.

parameter estimation algorithm converges, often in less than 10 iterations, to pdfs which represent the local statistics of real data well. The new speckle filter seems to perform well, it smooths homogeneous regions, while at the same time preserve important details in heterogeneous regions.

REFERENCES

[1] R. F. Wagner, S. W. Smith, J. M. Sandrik, and H. Lopez, "Statistics of speckle in ultrasound B-scans," *IEEE Trans. Sonics Ultrason.*, vol. 30, no. 3, pp. 156-163, 1983.

[2] E. Jakeman and P. N. Pusey, "A model for non-Rayleigh sea echo", *IEEE Trans. Antennas Propag.*, vol. AP-31, no. 4, pp. 490-498, 1976.

[3] P. M. Shankar, J. M. Reid, H. Ortega, C. W. Piccoli, and B. B. Goldberg, "Use of non-Rayleigh statistics for the identification of tumors in ultrasonic B-scans of the breast," *IEEE Trans.Med. Imag.*, vol. 12, no. 4, pp. 687-692, 1993.

[4] P. M. Shankar, "A general statistical model for ultrasonic scattering from tissues," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 47, no. 3, pp. 727-736, 2000.

[5] P. M. Shankar,"A compound scattering pdf for the ultrasonic echo envelope and its relationship to K and Nakagami distributions," *IEEE Trans. Ultrason., Ferroelect., Freq. Contr.*, vol. 50, no. 3, pp. 339-343, 2003.

[6] T. Eltoft, "The Rician Inverse Gaussian Distribution: A new model for non-Rayleigh Signal Amplitude Statistics," *IEEE Trans. Image Process.*, vol. 14, no. 11, 2005.

[7] T. Eltoft, "Modeling the amplitude statistics of ultrasonic images," *IEEE Trans. Med. Imag.*, vol. 25, no. 2, 2006.

[8] A. F. Andrews and C. L. Mallow, "Scale mixtures of Normal Distributions," *Journal of the Royal Statistical Society. Series B*, vol. 36, no. 1, pp. 99-102, 1974.

[9] O. E. Barndorff-Nielsen, "Normal Inverse Gaussian Distributions and Stochastic Volatility Modeling," *Scand. J. Statist.*, vol. 24, pp. 1-13, 1997.

[10] X. Hao, S. Gao and and X. Gao, "A novel multiscale nonlinear thresholding method for ultrasonic speckle suppression," *IEEE Trans. Med. Imag.*, vol. 18, no. 9, pp. 787-794, 1999.

[11] T. Loupas, W. N. Mcdicken and P. L. Allan, "An adaptive weighted median filter for speckle suppression in medical ultrasonic images," *IEEE Trans. Circuits syst.*, vol. 36, no. 1, pp. 129-135, 1989.

[12] J. S. Lee, "Digital image enhancement and noise noise filtering by use of local statistics," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. PAMI-2, no. 2, pp. 165-168, 1980.

[13] D. T. Kuan, A. A. Sawchuck, T. C. Strand, and P. Chavel, "Adaptive Noise Smoothing Filter for Images with Signal Dependent Noise," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 7, pp. 165-177, 1985.

[14] A. Lopes, E. Nezry, R. Touzi, and H. Laur, "Maximum a posteriori filtering and first order texture models in SAR images,"in *Proc. IGARSS 1990*, Washington D. C., May 1990, pp. 2409-2412.