14th European Signal Processing Conference (EUSIPCO 2006), Florence, Italy, September 4-8, 2006, copyright by EURASIP

GLRT-BASED DETECTION-ESTIMATION OF GAUSSIAN SIGNALS IN UNDER-SAMPLED TRAINING CONDITIONS

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ABSTRACT

A likelihood ratio test has recently been developed for detection-estimation in under-sampled scenarios where the number of training data T is less than the number of antenna elements M. This test can be applied in a GLRT detection-estimation framework to many problems not satisfactorily addressed by conventional techniques. In particular, we consider under-sampled MUSIC "performance breakdown" phenomenon for independent sources, and use the under-sampled likelihood ratio to detect the presence of MUSIC outliers.

1. INTRODUCTION

Recently, it has been demonstrated that the GLRT-based detection-estimation approach may be successfully used for scenarios that are poorly addressed by conventional detection-estimation techniques [1], [2], [3]. Specifically, in [1], it was shown that the GLRT framework is instrumental to overcome the well-known MUSIC-specific "performance breakdown" phenomenon, where under certain conditions MUSIC starts to generate severely erroneous direction of arrival (DoA) estimates. In [4], a particular GLRT-based iterative scheme was suggested to address circular, and in fact, arbitrary antenna array geometries to provide MUSIC-specific performance breakdown "prediction and cure" of such DoA "outliers" [5],[6],[7]. With all these prior developments, the training sample volume T was considered to exceed the dimension M of the adaptive filter (antenna array) so that conventional (non-degenerate) ML estimates of the covariance matrix exists.

Yet in many practical applications the number of independent identically distributed (i.i.d) training samples need not approach this conventional ("Wishart" [8]) training condition ($T > M$). One of the well-known families of this kind is the one with low signal subspace dimension, where the number m of the covariance matrix eigenvalues much greater than the minimal eigenvalue (equal to ambient white noise power) is small relative to the matrix dimension M ($m < M$). In this general case an admissible covariance matrix could be introduced in the form

$$R = \sigma_0^2 I_M + R_S;\;\; R_S = \mathscr{U}_m \Lambda_0 \mathscr{U}_m^H;\;\; \Lambda_0 = \Lambda_m - \sigma_0^2 I_m, \qquad (1)$$

where $\mathscr{U}_m \in C^{M\times m}$ and $\Lambda_m \in R_+^{m\times m}$ are the $(M\times m)$-variate and $(m\times m)$-variate matrices of "signal subspace" eigenvectors and (positive) eigenvalues respectively. For localization of the "signal subspace" of such a "low rank" covariance matrix, the sample support (i.e. the number of independent identically distributed training samples) need only equal or be greater than m rather than M.

While DOA estimation techniques exist for under-sampled ($T < M$) training conditions, the modern GLRT-based detection-estimation techniques do not embrace this scenario, mainly because proper likelihood ratios have not been yet introduced.

In [9] we have suggested an LR for under-sampled ($T < M$) scenarios that has all the required properties for use in a GLRT framework. Formulation of that LR is briefly revisited in Section 2 below. In section 3, we utilize the undersampled LR in a "prediction and cure" methodology in the presence of MUSIC performance breakdown. And in section 4, we provide simulation results for a particular MUSIC performance breakdown example.

2. LIKELIHOOD RATIO FOR UNDER-SAMPLED GAUSSIAN SCENARIO FOR GLRT-BASED DETECTION-ESTIMATION

The central idea of the suggested [1],[2],[3],[4] GLRT-based detection-estimation approach introduced for "properly" sampled ($T \geqslant M$) training conditions is based on the property of the likelihood function for the stochastic complex Gaussian model.

Indeed, let $X_T = [x_1, \ldots, x_T]$, $x_j \in \mathscr{CN}(0, R_0)$. Then the likelihood function w.r.t. any $R > 0$ exists and is non-degenerate even for under-sampled training conditions ($T < M$).

$$\mathscr{L}(X_T, R) = \left[\frac{1}{\pi \det R} \exp\{-\mathrm{Tr}\,[R^{-1}\hat{R}]\}\right]^T \qquad (2)$$

where

$$\hat{R} = \frac{1}{T}\sum_{j=1}^{T} x_j x_j^H. \qquad (3)$$

For "properly" sampled ($T \geqslant M$) conditions, when $\det \hat{R} > 0$ (with probability one), instead of the likelihood function (2), one can consider the likelihood ratio

$$LR(R) = \frac{\mathscr{L}(X_T, R)}{\max_R \mathscr{L}(X_T, R)} \qquad (4)$$

where

$$\max_R \mathscr{L}(X_T, R) = \left[\frac{\exp(-M)}{\pi \det \hat{R}}\right]^T, \;\text{ for } R = \hat{R} \qquad (5)$$

and therefore

$$LR(R) = \left[\frac{\det R^{-1}\hat{R} \exp M}{\exp\{\mathrm{Tr}\,[R^{-1}\hat{R}]\}}\right]^T \leqslant 1 \qquad (6)$$

is equal to one for the generic ML covariance matrix estimate $\hat{R}$.

The most important property of this $LR(R)$ is that for the actual (true) covariance matrix $R = R_0$, the p.d.f. of $LR(R_0)$ *does not depend on* R_0, since

$$LR(R_0) = \left[\frac{\det \hat{C}_0 \exp M}{\exp\{\mathrm{Tr}\,\hat{C}_0\}}\right]^T \qquad (7)$$

Work funded under DSTO/RLM R&D CA 290905

where $\hat{C}_0 = R_0^{-\frac{1}{2}} \hat{R} R_0^{-\frac{1}{2}}$, and for $T \geqslant M$, $\hat{C}_0 \sim \mathscr{CW}(T \geqslant M, M, I_M)$, i.e. $\hat{C}$ is described by the scenario-free complex Wishart distribution, fully specified by the parameters M and T [10].

Quite a straight-forward observation

$$\max_{R \in \mathscr{R}} LR(R) > LR(R_0), \;\; R_0 \in \mathscr{R}, \tag{8}$$

where $\mathscr{R}$ is the admissible set of covariance matrices that, obviously, includes R_0, along with the scenario-free property for the p.d.f. for $LR(R_0)$, form the basis of the introduced [1],[2] GLRT-based detection-estimation scheme. This means that the GLRT-based scheme finds solutions that are statistically as "likely" as the true covariance matrix R_0, without any *a priori* or clairvoyant knowledge of that true solution.

For $T < M$, $\det \hat{R} = 0$ and the $LR(R)$ can no longer be constructed as in (6). Therefore, in order to expand GLRT-based detection-estimation methodology over the undersampled ($T < M$) scenario, in [9] we had to develop an $LR_u(R)$ that meets the following conditions:

a) **Normalization condition**:

$$0 < LR_u(R) \leqslant 1 \tag{9}$$

b) **Transition behavior**: $LR_u(R)$ should be an "analytic extension" of the $LR(R)$ (6), i.e.

$$LR_u(R) = LR(R) \;\; \text{for } T \geqslant M \tag{10}$$

c) **Invariance property**:

$$p.d.f\,[LR_u(R_0)] = f(M, T). \tag{11}$$

Such a $LR_u(R)$ that meets the above requirements was derived in [9], based on the observation that rank-m "signal-subspace" matrix R_S in the most general case is fully specified by its first m rows (columns), which means that the number of real-valued degrees of freedom (rDOF) required for exhaustive description of R in (1) is

$$rDOF(\hat{R}) = 1 + 2Mm - m^2. \tag{12}$$

On the other hand, for $T < M$, the sample covariance matrix $\hat{R}$ in (3) is fully specified by its first T columns (rows), which means

$$rDOF(\hat{R}) = 2MT - T^2 \tag{13}$$

Therefore, for $T > m$, functionally independent entries in $\hat{R}$ can serve as sufficient statistics for estimation of the "low-rank" covariance matrix R in (1), and the minimum number of such elements is specified by (12). In [9], we proposed to consider for $m < T < M$ a $(2T-1)$ wide band of the matrix $\hat{R}$:

$$\Omega^{\hat{R}} : \left[\hat{r}_{ij}\right] \quad |i-j| \leqslant T-1; \quad \hat{R} = \left[\hat{r}_{ij}\right] \; i,j = 1, \ldots M. \tag{14}$$

Note that the number of real-valued degrees of freedom for this band is equal to

$$DOF(\hat{R}_{B(T)}) = 2MT - T^2 - (M-T) \tag{15}$$

and is only $(M-T)$ degrees short from the $(2MT - T^2)$ degrees of freedom in $\hat{R}$.

Specifically, we suggest to consider the $(2T-1)$-wide band of the "pre-whitened" matrix $\hat{C} = R^{-\frac{1}{2}} \hat{R} R^{-\frac{1}{2}}$, where R is a p.d. Hermitian covariance matrix model. Since this band does not uniquely specify the entire matrix $\hat{C}$ (rank $\hat{C} = T$), a number of completions of the band $[\hat{c}_{ij}], |i-j| \leqslant T-1$ exist (including the original matrix $\hat{C}$). We will use the completion $\hat{C}^{(p)}$ with the maximal (non-zero) determinant, specified by the Dym-Gohberg band-extension method [11], [12].

In [12],[13] it was proven that amongst all band extensions, the Dym-Gohberg extension has the maximal determinant, given by

$$\det[\hat{C}^{(p)}]^{-1} = \prod_{q=1}^{M} e_q^T \hat{C}_q^{-1} e_q \tag{16}$$

where e_q is a column vector of length M with a single unity entry at position q and $\hat{C}_q$ is the $(L(q)-q+1) \times (L(q)-q+1)$ Hermitian central block matrix in $\hat{C}$ with

$$\hat{C}_q = \begin{bmatrix} \hat{c}_{q,q} & \cdots & \hat{c}_{q,L(q)} \\ \vdots & & \vdots \\ \hat{c}_{L(q),q} & \cdots & \hat{c}_{L(q),L(q)} \end{bmatrix} \tag{17}$$

for $q = 1, \ldots, M$, $p \leqslant T-1$, and $L(q) = min\{M, q+p\}$.

The Dym-Gohberg band extension method, applied to the rank-deficient sample matrix $\hat{C}$, transforms this matrix into a positive definite Hermitian matrix $\hat{C}^{(p)}$ with exactly the same elements as the sample matrix $\hat{C}$ within the $(2p+1)$-wide diagonal band.

Moreover, this p.d. matrix $\hat{C}^{(p)}$ *is uniquely specified* by all different $(p+1)$-variate central block matrices in $\hat{C}$, and the only necessary and sufficient condition for such transformations to exist, is the positive definiteness of all $(p+1)$-variate submatrices $\hat{C}_q$ in $\hat{C}$. Let $m < p \leqslant T-1$, when we have $DOF(R_S) < DOF(\hat{C}^{(p)})$. For this reason, we introduce the following likelihood ratio $\Lambda_0^{(p)}(R)$ for our under-sampled scenario:

$$\Lambda_0^{(p)}(R) = \left[\frac{\det[(R^{-\frac{1}{2}} \hat{R} R^{-\frac{1}{2}})^{(p)}] \exp M}{\exp\{\operatorname{Tr} \hat{R} R^{-1}\}}\right]^{\frac{1}{M}} \leqslant 1 \tag{18}$$

since $\operatorname{Tr} \hat{R}R^{-1} = \operatorname{Tr}\,[(R^{-\frac{1}{2}} \hat{R} R^{-\frac{1}{2}})^{(p)}]$. Here $(R^{-\frac{1}{2}} \hat{R} R^{-\frac{1}{2}})^{(p)} = \hat{C}^{(p)}$ is the Dym-Gohberg p-band transformation of the matrix $\hat{C}$. Note that in fact, $\Lambda_0^{(p)}(R)$ calculation does not require actual reconstruction of the Dym-Gohberg extension, since it's determinant is explicitly calculated via block-matrix $\hat{C}_q$ in (16). In this regard, this likelihood ratio may be treated as a test on $\mathscr{E}\{\hat{C}_q\} = I_{Lq-q+1}$, simultaneously for all q, which for $m < p$ is consistent with the original testing problem $\mathscr{E}\{\hat{C}\} = I_M$.

For $R = R_0$ we get $\hat{C} = \hat{C}_0$, where $\hat{C}_0 \sim \mathscr{ACW}(T < M, M, I_M)$, i.e. $\hat{C}_0$ is now described by the scenario-free anti-Wishart complex distribution, specified by T and M only [8]. Therefore, the p.d.f. for the $\Lambda_0^{(p)}(R_0)$ does not depend on scenario, and must be specified by the parameters M, T and p only. Indeed, according to Theorem 2 in [9], for the actual (true) covariance matrix R_0, the LR (18) is a random value that is statistically equivalent to the following representation:

$$\Lambda_0^{(p)}(R_0) = \exp 1 \left[\prod_{q=1}^{M} \Omega_q \varphi_q\right]^{\frac{1}{M}} \tag{19}$$

$$\varphi_q \sim \frac{\varphi_q^{(T-\nu-1)}(1-\varphi_q)^{(\nu-1)}}{B[\nu, T-\nu]}, \quad 1 \leqslant \nu \equiv L(q) - q < p$$

$$L(q) = \min\{M, q+p\}$$

$$\Omega_q \sim \frac{C_{qq}}{T} \exp\left[-\frac{C_{qq}}{T}\right] \quad C_{qq} \sim \frac{C_{qq}^{T-1}}{\Gamma(T)} \exp(-C_{qq})$$

where φ_q and Ω_q are independent and $B[\nu, T-\nu]$ and $\Gamma(T)$ are the incomplete beta and gamma functions, respectively. The p.d.f. for

$\Lambda_0^{(p)}(R_0)$ is therefore independent on scenario, and is fully specified by M,T, and p.

Now the GLRT-based adaptive detection-estimation framework has been made available to embrace "difficult" under-sampled scenarios and in what follows we illustrate its efficiency in an important example.

3. "UNDER-SAMPLED" PERFORMANCE BREAKDOWN "PREDICTION AND CURE" FOR SUBSPACE-BASED DETECTION-ESTIMATION TECHNIQUES

Let us consider the M-element antenna array, designed to resolve up to $m_{max} < M$ sources, with the training support of T training samples $X_T = [x_1, \dots, x_T]$ (i.e. $m_{max} < T < M$).

According to the GLRT detection-estimation methodology, for $\mu = 0, 1, \dots, m_{max}$ we have to generate under-sampled maximum likelihood (USML) models $\hat{R}_\mu$:

$$\hat{R}_\mu = \hat{\sigma}_0^2 I + \mathscr{S}_\mu(\hat{\theta}_\mu) B_\mu \mathscr{S}_\mu(\hat{\theta}_\mu) \tag{20}$$

where

$$\hat{R}_\mu = \arg\max_{R_\mu} \Lambda_0^{(p)}(R_\mu); \;\; m_{max} < p < T \tag{21}$$

and the smallest μ where $\Lambda_0^{(p)}(R_\mu)$ exceeds the pre-calculated threshold ϑ_{FA}, is treated as the estimate $\hat{m}$ for the number of sources m:

$$\hat{m} = \arg\min_{\mu} \Lambda_0^{(p)}(\hat{R}_\mu) \geqslant \vartheta_{FA} \tag{22}$$

Here $\mathscr{S}_\mu(\theta_\mu)$ is the $[M \times \mu]$-variate antenna "manifold" matrix, uniquely specified by a set of μ parameters (DOA's) $\theta_\mu = [\theta_1, \dots, \theta_\mu]$, B_μ is the $(\mu \times \mu)$-variate Hermitian non-negative definite (n.n.d.) inter-source covariance matrix, and σ_0^2 is the additive white noise power.

Since

$$\max_{\mu \geqslant m} \Lambda_0^{(p)}(R_\mu) \geqslant \Lambda_0^{(p)}(R_0), \tag{23}$$

the scenario-free p.d.f. for $\Lambda_0^{(p)}(R_0)$ could be used to calculate the threshold ϑ_{FA} (22) for the lower bound of the given probability of false alarm P_{FA}.

$$\int_{\vartheta_{FA}}^{1} f\left[\Lambda_0^{(p)}(R_0)\right] d\Lambda_0^{(p)} = P_{FA} \tag{24}$$

An analytic expression for this p.d.f could be given, but it is cumbersome to calculate and as an alternative, direct Monte-Carlo simulations of (19) may be employed for a given M, T and p to pre-calculate the ϑ_{FA}.

Depending on the problem at hand, this quite generic GLRT-based detection-estimation framework could be now adopted for under-sampled scenarios, with appropriate techniques used for LR maximization in (21). Specifically, it now can be used for subspace-specific "performance breakdown" "prediction and cure" in a similar way to the method suggested in [4],[14] for conventional ($T > M$) training conditions.

According to this methodology, the entire procedure consists of the following five steps.

Step 1 "Breakdown Prediction".
The covariance matrix model $R_{\hat{m}}^{M\nu}$ conventionally generated by, say, the Wax-Kailath detection algorithm, followed by MUSIC, is tested by the inequality (22). If the threshold in (22) is exceeded, then the solution $R_{\hat{m}}^{M\nu}$ is accepted in terms of the LR being statistically as good as the true parameters that specify the covariance matrix R_0. On the other hand, the presence of MUSIC-specific outliers are expected to be "predicted" when the respective model do not reach the threshold.

Step 2 "Local refinement".
If MUSIC-generated estimates are within a convex proximity to the "proper" solution, local optimization by the Gauss-Newton or Neder-Mead (for example) algorithm can deliver a solution. Otherwise, when a severe outlier is present within the DOA set $\{\theta_{\hat{m}}\}$, such local optimization fails.

Step 3 "Outlier identification".
Dealing with identifiable scenarios, we have to assume that the LR threshold is not achieved due to some missing DOA estimate(s). Therefore, the source in the model (20) which can be deleted from the model with the minimal degradation in LR, is treated as an outlier.

Step 4 "Outlier replacement".
Instead of the "outlier" excluded at step 3, we now search for the source with DOA estimate that maximally contributes to the LR. 1-D MUSIC type search for this maximum could be used.

Step 5 "Final refinement".
Local optimization, as per step 2, is executed in the vicinity of the new set of DOA's.

If the original set includes more than a single outlier, and as a result the threshold is not exceeded, the procedure is repeated until the threshold is exceeded. However with at least probability P_{FA}, this procedure may not reach the threshold and must be terminated at some stage.

In [1],[3],[4] this technique was illustrated for uniform linear and circular antenna arrays under "conventional" training conditions with independent Gaussian sources. In Section 4, we provide simulation results that illustrate efficiency of this approach for under-sampled training conditions and both independent and coherent Gaussian sources.

The introduced GLRT detection-estimation framework allows introduction of "prediction and cure" for MUSIC-specific performance breakdown conditions.

4. SIMULATION RESULTS

Let us consider a scenario with M=10 antenna elements in a uniform line array and digital receiver-per-element architecture. Figure 1 shows the mean results of likelihood ratio formation for various levels of training data support. The three key properties of the suggested under-sampled likelihood ratio (18) can be seen in Figure 1. The LR is normalised between 0 and 1, it transitions properly from the under-sampled likelihood ratio to a standard likelihood ratio at $T = M$, and the analytically derived LR mean (see [9] for a derivation), which is by definition scenario-free, agrees with both the clairvoyant solution and MUSIC-derived solution.

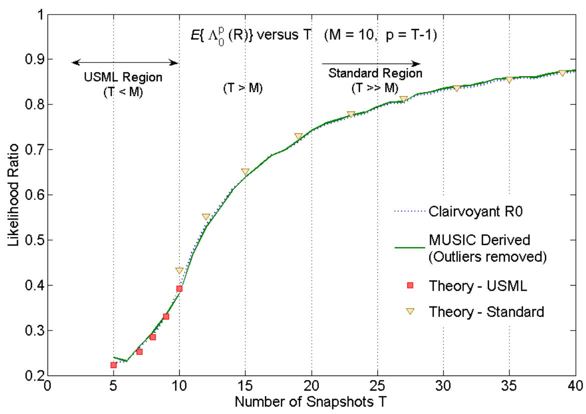

Figure 1: Theoretical and Observed LRs

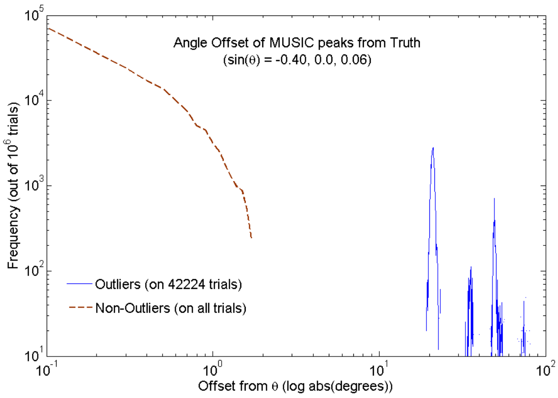

Figure 2: Observed Angle Errors of MUSIC DOAs

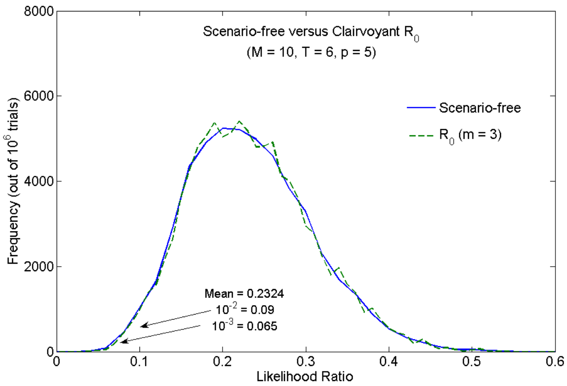

Figure 3: Pre-Calculable LR PDF Matches Clairvoyant R_0

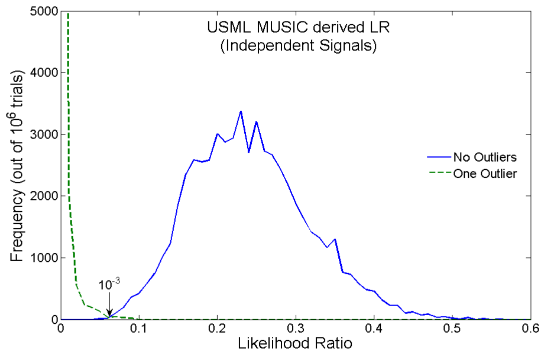

Figure 4: LR PDFs of Uncorrelated Signals Scenario

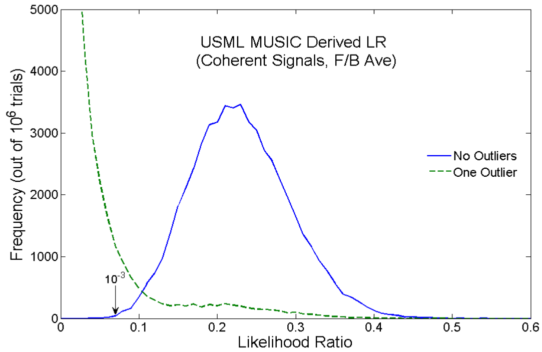

Figure 5: LR PDFs of Correlated Signals Scenario

The level of training support is set to an under-sampled level ($T = 6$) to determine whether the USML LR can be used to accurately detect the presence of MUSIC outliers, as the first step in a GLRT-based rectification methodology. For $d/\lambda = 0.5$, we consider a three source scenarios with independent sources with an input SNR of 20dB per source:

$$\sin(\theta_S) = \{-.40, 0.0, 0.06\} \qquad (25)$$

To generate a "difficult" circumstance, we have selected the third source separation to reside within the MUSIC performance breakdown region. Specifically, for the selected scenario, MUSIC provided 42.2% severely erroneous DOA estimates ("outliers"). The distribution of the MUSIC generated outliers can be seen in Figure 2. Based on this angular distribution, a value of $\pm 2.0^o$ was used as an association window size with the true signal DOAs while determining whether each trial containing an outlier.

Results of our GLRT-based scheme that adopts the under-sampled LR (18), are shown in Figures 3 and 4 and are summarised by Tables 1-3. Figure 3 show that the pre-calculated p.d.f for $\Lambda_0^{(p)}(R_0)$ (which is scenario-free) agrees well with the exhibited clairvoyant R_0 seen during the Monte-Carlo trials. Figure 4 show that the p.d.f's of the "outlier" and "non-outlier" p.d.f.s are well separated and therefore can be properly classified with a thresholding step.

In Table 1, we adopted a threshold calculated for a $P_{FA} = 10^{-3}$, to assess "practical" non-clairvoyant performance of our routine. Let us emphasize that $p = T - 1$ means that only M=5 element antenna covariance array subsets are involved in model $\hat{R}^{(p)}$ reconstruction, yet quite efficient performance is demonstrated here without any diagonal loading or use of other *a-priori* information.

As previously suggested by the well separated p.d.f.s in Figure 4, Step 1 of Table 1 shows that around very few non-outlier trials were misclassified (as expected based on the use of a $P_{FA} = 10^{-3}$ threshold). Subsequent steps in the GLRT "prediction and cure" methodology show that virtually all MUSIC-specific "outliers" can be rectified.

The introduced outlier rectification scheme may also be applied for scenarios with fully correlated sources. There are no modifications to the p.d.f pre-calculation, since it is scenario-free. For uniform linear antenna arrays, "forward-backward" spatial smoothing for each training sample is typically used to provide an M_α-variate sample covariance matrix ($M_\alpha < (M - m_{max}/2)$) this is used for conventional detection-estimation [15]. In that case, dependence on T is less critical and in most cases, under-sampled training conditions ($T < M$ or even $T < m_{max}$) are in use. Also, computationally, the GLRT routines must be modified slightly to provide optimization across a complex-valued (rank 1) inter-source covariance matrix rather than a real, positive valued diagonal inter-source covariance matrix. Results for a fully coherent 3 source scenario with the same locations given in (25) are summarized by Figure 5 and Tables 2-3.

Figure 5 show that the p.d.f's of the "outlier" and "non-outlier" p.d.f.s overlap more than in the uncorrelated signal case and therefore are not as well classified with a thresholding step. The results for the fully coherent signal scenario show that improvement can be ultimately achieved via the GLRT-based outlier rectification scheme, but some trials with outliers result in a model LR which exceeds the threshold significantly and becoming indistinguishable

Table 1: "Practical Threshold" - Independent Signals

GLRT Step	Outlier Detected	"Truth"	Mean LR	$P_{FA} = 10^{-3}$ $\alpha = 0.065$
1. Breakdown	No	56.3%	0.2451	56.3%
Prediction	Yes	43.7%	0.0015	43.7%
2. Local	No	68.2%	0.2382	72.4%
Refinement	Yes	31.8%	0.0218	27.6%
3/4. Outlier	No	95.5%	0.2295	99.0%
Predict/Cure	Yes	4.5%	0.1175	1.0%
4. Final	No	95.5%	0.2297	99.2%
Refinement	Yes	4.5%	0.1175	0.8%

in an LR sense from trials without outliers. This is an example of the so-called "maximum-likelihood performance breakdown phenomenon" [1]. Obviously, if a particular model R_μ is close enough to such a ML breakdown condition, local refinement at Step 2 can drive it above the threshold, despite an "outlier" being present in R_μ. It is then excluded from further rectification since it is classified (incorrectly) as outlier-free. Therefore, only the "gap" between MUSIC-specific and maximum likelihood performance breakdown conditions may be rectified by the suggested GLRT-based technique. While significantly better performance in this case can be achieved by avoiding the local LR optimization step (Step 2) prior to "outlier prediction and cure" (see Table 3), the ML breakdown condition still prevents complete rectification.

Table 2: "Practical Threshold" - Coherent Signals Scenario

GLRT Step	Outlier Detected	"Truth"	Mean LR	$P_{FA} = 10^{-3}$ $\alpha = 0.065$
1. Breakdown	No	55.5%	0.2294	61.8%
Prediction	Yes	44.5%	0.0346	38.2%
2. Local	No	60.9%	0.2302	86.6%
Refinement	Yes	39.1%	0.1132	13.4%
3/4. Outlier	No	62.7%	0.2280	98.6%
Predict/Cure	Yes	37.3%	0.1321	1.4%
4. Final	No	63.7%	0.2265	99.7%
Refinement	Yes	36.3%	0.1347	0.3%

Table 3: No Local Refinement - Coherent Signal Scenario

GLRT Step	Outlier Detected	"Truth"	Mean LR	$P_{FA} = 10^{-3}$ $\alpha = 0.065$
1. Breakdown	No	55.5%	0.2294	61.8%
Prediction	Yes	45.5%	0.0346	38.2%
3/4. Outlier	No	93.6%	0.2268	100%
Predict/Cure	Yes	6.4%	0.1464	0%
4. Final	No	93.6%	0.2268	100%
Refinement	Yes	6.4%	0.1464	0%

5. SUMMARY AND CONCLUSION

We have shown that the LR test for under-sampled conditions introduced in [9] can be used to demonstrate significant improvement in detection-estimation performance within a MUSIC-specific breakdown threshold area. Specifically, for scenarios with either independent or fully coherent Gaussian sources, we demonstrated capabilities of our GLRT-based detection-estimation rectification scheme to recover the majority of severely erroneous solutions (outliers) produced by conventional MUSIC in ˜45% of trials in a particular scenario. The previously introduced GLRT-based detection-estimation methodology is now extended to embrace the practically important class of under-sampled training conditions.

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