NO MINIMUM RATE MULTISAMPLING OF A FOURIER SERIES

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ABSTRACT

I examine the possibility of sampling a Fourier series with multiple, uniform rates that are not required to be larger than any particular frequency. This is allowed because convolution of a Fourier series with a train of delta functions in the Fourier domain causes overlap in the Fourier domain only in isolated cases. Furthermore, I can restrict this overlap to not occur in more than one sampled transform. I use three different sampling rates, not required to be greater than any particular frequency, yet satisfying certain irrational relationships, which I specify. The three separate Fourier domains from each rate are compared, and a filter is used which outputs only those terms which are common to all three. In some cases, it might be necessary to introduce a fourth sampling rate. The result is that the original Fourier series is obtained from the filter and inverse transform.

1. INTRODUCTION

In recent years, there has been a considerable amount of research devoted to the idea of sampling a signal at a rate less than the Nyquist rate [1]–[5]. In these cases, sampling at such a rate is a result of gaps in the frequency occupancies of the signal in the Fourier domain.

There has also been a limited amount of attention given to sampling periodic signals. One paper [6] does suggest a double-sampling technique whereby an appropriate ratio of sampling frequencies yields the ability to detect aliased frequencies in periodic signals. The bandwidth of the combination of the two sampling frequencies is a function of their least common multiple. Another paper [7] shows that sampling periodic piecewise polynomial signals with a finite number of degrees of freedom can be sampled at a sub-Nyquist rate.

Very little attention, however, has been devoted to what may be the most accurate representation of a real signal – the Fourier series. Representation of a signal as a Fourier series does have certain requirements. The signal must be periodic, it must be continuous, and it must be piecewise smooth, which means that its derivative must exist everywhere, except at isolated points. The triangle wave is an example of a continuous, piecewise smooth function while the square wave is not. Despite these requirements, the Fourier series fits a large class of periodic signals.

In the Fourier domain, a Fourier series appears as a series of delta functions. Since a delta function by definition has no width, it seemed as though it might be possible to sample it with no minimum requirement.

Inherent to a Fourier series is the idea of rationality. The sine and cosine terms all have frequencies that are integer multiples of a base frequency. Because of this, any one frequency in any of the terms in the series is a rational multiple of any of the other frequencies. It came as no surprise, then, that the "trick" to sampling a Fourier series with low rates would involve irrational relationships.

2. SAMPLING OF A FOURIER SERIES

We begin by representing a signal as a Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1,2,3...}^{\infty} a_n \cos(\omega_n t) + \sum_{n=1,2,3...}^{\infty} b_n \sin(\omega_n t)\,.$$

Recall especially that the frequencies shown in the sine and cosine functions are all integer multiples of some base frequency.

We will now sample our signal with three separate rates

$$S_{0,1,2}(t) = f(t) \sum_{m=-\infty}^{\infty} \delta(t - mT_{0,1,2})\,.$$

Then we take the Fourier transform:

$$S_{0,1,2}(\omega) = F(\omega) \otimes \frac{\sqrt{2\pi}}{T_{0,1,2}} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_{s_{0,1,2}})$$

where $\omega_{s_{0,1,2}} = 2\pi / T_{0,1,2}$.

Under convolution, the transform becomes:

$$S_{0,1,2}(\omega) = \frac{a_0\sqrt{2\pi}}{2T_{0,1,2}} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_{s_{0,1,2}})$$

$$+\frac{1}{T_{0,1,2}}\sqrt{\frac{\pi}{2}}\sum_{n=1,2,3...}^{\infty}\sum_{m=-\infty}^{\infty}a_n(\delta(\omega-m\omega_{s_{0,1,2}}-\omega_n)+\delta(\omega-m\omega_{s_{0,1,2}}+\omega_n))$$
$$+\frac{i}{T_{0,1,2}}\sqrt{\frac{\pi}{2}}\sum_{n=1,2,3...}^{\infty}\sum_{m=-\infty}^{\infty}b_n(\delta(\omega-m\omega_{s_{0,1,2}}+\omega_n)-\delta(\omega-m\omega_{s_{0,1,2}}-\omega_n)) . \quad (1)$$

Notice that the m=0 terms all represent the actual sampled function. If we eliminate all the other terms, the inverse transform would yield our original signal. Our goal is to pick the sampling frequencies such that only the m=0 terms are common in frequency to all three transforms. We will then implement a filter that removes any terms that are not exactly shared between the three transforms.

3. CHOOSING THE SAMPLING FREQUENCIES

Our goal, now, is to choose our sampling frequencies such that only the m=0 terms in (1) are shared by all three transforms. There are three ways in which the terms in all three transforms can be shared. The first way is that the m=0 or "true" terms all line up. This will happen regardless of our selection of sampling frequencies. The second is that other selections of m will cause what I will call "cross-aliasing" to occur. For example, selecting m=24 in one term in one transform and selecting m=50 in another transform could cause the delta functions to occur in exactly the same place in both transforms, if the sampling frequencies are chosen with the proper relationship. We will completely eliminate the possibility of this occurring in all three transforms.

The third manner in which the delta functions can be shared in the transforms is the result of an "unlucky" selection of a sampling frequency. These "unlucky" selections do not cause the frequency occupancies of the sampled functions from (1) to change. They do, however, distort the amplitudes of coefficients to the delta functions, and thus, make re-creating the function from our filter rather difficult. However, it will be possible to still re-create our function. We will refer to the third manner of sharing terms as "self-aliasing".

3.1 Cross-Aliasing

Recall, in (1), that the ω_n terms are all integer multiples of some base frequency. Thus, in order for cross aliasing to occur from all three transforms we will need

$$m\omega_{s_0}+M\omega_0=l\omega_{s_1}+L\omega_0=n\omega_{s_2}+N\omega_0 \quad (2)$$

where m, l, and n are integers not equal to zero; M, L, and N are integers; and ω_0 is the fundamental frequency of the Fourier series.

Creating separate equations we have

$$m\omega_{s_0}-l\omega_{s_1}=L\omega_0-M\omega_0$$

as well as

$$m\omega_{s_0}-n\omega_{s_2}=N\omega_0-M\omega_0 .$$

We can introduce a new integer into the RHS of both equations that replaces the two integers. Then we have

$$m\omega_{s_0}-l\omega_{s_1}=P\omega_0 \quad (3)$$

as well as

$$m\omega_{s_0}-n\omega_{s_2}=Q\omega_0 . \quad (4)$$

And now we divide (3) by (4):

$$\frac{m\omega_{s_0}-l\omega_{s_1}}{m\omega_{s_0}-n\omega_{s_2}}=\frac{P}{Q} . \quad (5)$$

This is equivalent to saying that the LHS of this equation must be a rational number in order for all three transforms to share cross-aliasing terms. In other words, if the LHS is irrational, we will not be able to satisfy the RHS, and all three transforms will not share cross-aliasing terms.

3.2 Self-Aliasing

We can always be "unlucky" and choose a sampling rate that is a rational multiple of the fundamental frequency in the Fourier series. This might distort the amplitudes of terms corresponding to the m=0 terms, if there are non-zero coefficients to the delta functions While we cannot eliminate this from happening in one of the transforms from (1), we can eliminate this from occurring in more than one transform.

For self aliasing to occur in two transforms we must have

$$m\omega_{s_0}=M\omega_0 \quad (6)$$

and

$$l\omega_{s_1}=L\omega_0 . \quad (7)$$

The solution is simple: we divide (6) by (7) and find that the ratio of the two sampling rates must be a rational number in order for self-aliasing to occur in both transforms. Therefore, if all three sampling rates are irrational multiples of each other, then self-aliasing will occur in only one transform.

Note that we have not made any restrictions on the *size* of the sampling frequencies. We do not need to make any. Our restrictions ensure that we will be able to "tell" what is an aliased frequency and what is not.

4. RECOVERY FILTER

We must still recover our original Fourier series from the three separate transforms. I give the filter as

$$F(\omega) = T_0 S_0(\omega) \int_{-\infty}^{\infty} \delta\,(\lim_{\varepsilon\to 0} \int_{\omega'''-\varepsilon}^{\omega'''+\varepsilon} S_1(\omega')d\omega'$$
$$- \lim_{\varepsilon\to 0} \int_{\omega'''-\varepsilon}^{\omega'''+\varepsilon} S_2(\omega'')d\omega'')\delta\,(\omega'''-\omega)\,d\omega'''. \quad (8)$$

This integral can be interpreted as deconvolution. The sampled functions can be interchanged as long as the function outside the integral is accompanied by its correct sampling period. The effect of the integral is to "select" those delta functions which are exactly shared between the two sampled functions within the integral. The delta functions must occur at the same frequency, and with the same amplitude. This effect, combined with the sampled function outside the integral, effectively selects those delta functions that are shared between the three sampled transforms.

But what if self-aliasing has occurred? If this is the case, then we will get different answers if we interchange S_0, S_1, and S_2. The reason is that self-aliasing only occurs in one of the transforms. If the self-aliased transform is outside the integral, then the self-aliased frequencies will correspond to a non-zero solution. If, however, the self-aliased transform is inside the integral, the self-aliased frequencies will correspond to a zero solution, since the amplitude of the delta function will not be exactly shared with the accompanying transform inside the integral.

The solution is to introduce a fourth sampling rate that satisfies our two irrational relationships. Finding the correct three out of the four transforms, then, is as simple as finding the three transforms that do not give us a different answers when we interchange them in (8). Using the solution from (8) as the function for the inverse Fourier transform, now, will give us our original series. The inverse transform, by the way, should be performed across all frequencies, from negative infinity to positive infinity.

5. CONCLUSION

I showed how a Fourier series can be sampled using three separate uniform rates that satisfy two irrational relationships, regardless of the size of the sampling rates. I explained that, if these relationships are followed, then the terms shared by all three resulting transforms will be a full representation of the Fourier series. Finally, I gave a filter which can be used to obtain the original series from the three transforms.

It should be noted that this paper is entirely theoretical. I did not attempt to sample a periodic function with irrational rates, mostly because irrational numbers do not work well in digital systems. If, however, irrational relationships can be approximated, then it may be possible to sample at very low rates with an adequate result. I leave this as an exercise for future researchers.

Using multiple samplers and approximating a periodic signal with a bandwidth greater than half the frequencies of these samplers is a problem that might be described as superresolution. While superresolution is most often used in the context of imaging, its applications could reach to other areas as well. Someday, perhaps, we will learn to extend ourselves beyond the bounds of classical sampling bandwidths, and intelligently explore those areas to which we were once blind.

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