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EFFICIENCY OF SUBSPACE-BASED ESTIMATORS

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ABSTRACT

This paper addresses subspace-based estimation and its purpose is to complement previously available theoretical results generally obtained for specific algorithms. We focus on asymptotically (in the number of measurements) minimum variance (AMV) estimators based on estimates of orthogonal projectors obtained from singular value decompositions of sample covariance matrices associated with the general linear model $\mathbf{y}_t = \mathbf{A}(\Theta)\mathbf{x}_t + \mathbf{n}_t$ where the signals $\mathbf{x}_t$ are complex circular or noncircular and dependent or independent. Using closed-form expressions of AMV bounds based on estimates of different orthogonal projectors, we prove that these AMV bounds attain the stochastic Cramer-Rao bound (CRB) in the case of independent circular or noncircular Gaussian signals.

1. INTRODUCTION

Subspace-based estimates, i.e., estimates obtained by exploiting the orthogonality between a sample subspace and a parameter-dependant subspace, have proved useful in many applications in signal processing. These methods have been applied successfully to a variety of problems, including array processing and linear system identification to estimate for example, directions-of-arrival (DOA) in narrowband array processing (see e.g., [1] and [2]) and finite impulse responses of single-input multiple-output (SIMO) channels (see e.g., [3]). There is considerable literature about the performance of such algorithms obtained in specific contexts, but few of them offer generic results. Among them, the work by Cardoso and Moulines [4] discusses the generalization of the optimal subspace fitting approach introduced by Ottersten *et al.* [5] in the DOA estimation context and shows the equivalence between subspace fitting and subspace matching.

The aim of this paper is to complement these generic results by extending the efficiency of the optimal DOA subspace fitting algorithms indirectly derived by comparison to the stochastic maximum likelihood estimator [5].

The paper is organized as follows. The generic signal model and a motivating example in the context of SIMO and SISO channels with noncircular inputs are given in Section 2. Section 3 applies the notion of AMV and asymptotically best consistent estimator (ABC) introduced by Porat and Friedlander [6] and Stoica *et al* [7] respectively, extends it to different projection-based statistics, and proves that these AMV bounds attain the stochastic Cramer-Rao bound (CRB) in the case of independent circular or noncircular Gaussian signals. Finally, Section 4 illustrates the AMV bound in the context of SISO channels with a BPSK input.

2. SIGNAL MODEL AND MOTIVATING EXAMPLE

In many applications, it is of interest to estimate the parameter $\Theta \in \mathbb{R}^K$ from the following M-variate complex valued wide sense stationary times series

$$\mathbf{y}_t = \mathbf{A}(\Theta)\mathbf{x}_t + \mathbf{n}_t, \quad t = 1,\dots,T, \tag{1}$$

where $\mathbf{A}(\Theta)\mathbf{x}_t$ and $\mathbf{n}_t$ model the signals of interest and additive measurement noise, respectively. In this general model, $\mathbf{x}_t$ and $\mathbf{n}_t$ are independent, zero-mean, second-order stationary time series. $(\mathbf{n}_t)_{t=1,\dots,T}$ are assumed Gaussian complex circular, independent and spatially uncorrelated with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma^2\mathbf{I}_M$, while $(\mathbf{x}_t)_{t=1,\dots,T}$ are complex noncircular, neither necessarily Gaussian nor independent, with $\mathbf{R}_x = \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^H)$ and $\mathbf{R}'_x = \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^T)$ nonsingular.

It is assumed that the $M \times L$ matrix $\mathbf{A}(\Theta)$ is deterministic and known as a function of the unknown signal parameters Θ. Of course, the probability distribution of $(\mathbf{y}_t)_{t=1,\dots,T}$ depends on extra parameters which are also unknown, but we are only interested here in the estimation of the parameter Θ. We suppose that $\mathrm{rank}(\mathbf{A}(\Theta)) = L < M$ and that Θ is uniquely[1] determined by the range space of $\mathbf{A}(\Theta)$. Consequently this leads to two covariance matrices of $\mathbf{y}_t$ that convey information about Θ:

$$\mathbf{R}_y = \mathbf{A}\mathbf{R}_x\mathbf{A}^H + \sigma^2\mathbf{I}_M \stackrel{\text{def}}{=} \mathbf{R}_s + \sigma^2\mathbf{I}_M$$

and

$$\mathbf{R}'_y = \mathbf{A}\mathbf{R}'_x\mathbf{A}^T \stackrel{\text{def}}{=} \mathbf{R}_{s'} \neq \mathbf{O}$$

and Θ is uniquely determined by the common projector $\mathbf{\Pi}_y$ onto the noise subspace associated with $\mathbf{R}_y$ and $\mathbf{R}'_y$ as well. Using the extended observation $\tilde{\mathbf{y}}_t \stackrel{\text{def}}{=} (\mathbf{y}_t^T, \mathbf{y}_t^H)^T$,

$$\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_t^H) = \tilde{\mathbf{A}}\mathbf{R}_{\tilde{x}}\tilde{\mathbf{A}}^H + \sigma^2\mathbf{I}_{2M} = \mathbf{R}_{\tilde{s}} + \sigma^2\mathbf{I}_{2M}$$

with

$$\tilde{\mathbf{A}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^* \end{pmatrix} \text{ and } \mathbf{R}_{\tilde{x}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{R}_x & \mathbf{R}'_x \\ \mathbf{R}'^*_x & \mathbf{R}^*_x \end{pmatrix},$$

where we suppose here that $\mathbf{R}_{\tilde{x}}$ is nonsingular, Θ is also determined by the orthogonal projector $\mathbf{\Pi}_{\tilde{y}}$ onto the $2L$-dimensional noise subspace of $\tilde{\mathbf{y}}_t$.

These covariance matrices are traditionally estimated by $\mathbf{R}_{y,T} = \frac{1}{T}\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t^H$, $\mathbf{R}'_{y,T} = \frac{1}{T}\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t^T$ and $\mathbf{R}_{\tilde{y},T} = \frac{1}{T}\sum_{t=1}^T \tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_t^H$, respectively. Thus, we can consider the orthogonal projectors $\mathbf{\Pi}_{y,T}$, $\mathbf{\Pi}'_{y,T}$ and $\mathbf{\Pi}_{\tilde{y},T}$ onto the noise subspace of the sample covariance matrices $\mathbf{R}_{y,T}$, $\mathbf{R}'_{y,T}$ and $\mathbf{R}_{\tilde{y},T}$ respectively. We note that there is not a one-to-one mapping between $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathbf{\Pi}_{\tilde{y},T}$, contrary to the one-to-one mapping $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T}) \longleftrightarrow \mathbf{R}_{\tilde{y},T}$.

[1] For system identification, note that a constraint on Θ must be added.

The first idea to estimate Θ from $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ is to use similar subspace-based algorithms derived from the projection matrices $\mathbf{\Pi}_{y,T}$ and $\mathbf{\Pi}'_{y,T}$. For example, for BPSK or OQPSK modulated inputs, the MUSIC-like algorithm that estimates the impulse response of a SIMO channel given in [3] from $\mathbf{\Pi}_{y,T}$ applies to $\mathbf{\Pi}'_{y,T}$ in the same way by replacing the EVD of $\mathbf{R}_{y,T}$ by the SVD of the complex symmetric matrix $\mathbf{R}'_{y,T}$. Consequently a problem crops up: how does one combine the statistics $\mathbf{\Pi}_{y,T}$ and $\mathbf{\Pi}'_{y,T}$ to improve the estimate of Θ? Is it possible to attain the AMV bound based on the statistic $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T})$?

Another idea to estimate Θ from $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ is to use subspace-based algorithms derived from the projection matrix $\mathbf{\Pi}_{\tilde{y},T}$. This approach has been proposed for the estimation of SIMO and SISO channels for BPSK or OQPSK modulated inputs in [8]. A question arises as well: Does there exist an algorithm based on the projector $\mathbf{\Pi}_{\tilde{y},T}$ whose performance approaches that of the AMV estimator based on $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$?

3. ASYMPTOTIC EFFICIENCY OF SUBSPACE-BASED AMV ESTIMATORS

3.1 Asymptotically minimum variance subspace-based estimator

A solution of the two aforementioned problems is to use the notion of AMV estimators based, respectively, on the matrix-valued statistics $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathbf{\Pi}_{\tilde{y},T}$. With this aim, we apply the following extension of the standard result [9] on AMV estimators to orthonormal projectors proved in [10]:

Lemma 1 *The covariance matrix $\mathbf{C}_\Theta$ of the asymptotic distribution of an estimator of Θ given by an arbitrary consistent subspace algorithm based on the statistics $\mathbf{\Pi}_{y,T}$, $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ or $\mathbf{\Pi}_{\tilde{y},T}$ is bounded below by the real symmetric matrix $\left(\mathscr{S}^H \mathbf{C}_s^{\#} \mathscr{S}\right)^{-1}$ where $\mathscr{S} \stackrel{\text{def}}{=} \frac{d\mathbf{s}(\Theta)}{d\Theta}$ with $\mathbf{s}(\Theta)$ is respectively* $\text{vec}(\mathbf{\Pi}_y)$, $\text{vec}(\mathbf{\Pi}_y, \mathbf{\Pi}'_y)$ *or* $\text{vec}((\mathbf{\Pi}_{\tilde{y}})$ *and where $\mathbf{C}_s$ is the singular first covariance matrix of the asymptotic distribution of the involved statistics.*

$$\mathbf{C}_\Theta \geq \left(\mathscr{S}^H \mathbf{C}_s^{\#} \mathscr{S}\right)^{-1}. \tag{2}$$

Following the proof given [10] in the case of independent observations for the DOA parametrization, the following expressions of $\mathbf{C}_s$ are obtained for the generic model (1) for dependent or independent observations.

Lemma 2 *The first covariance matrices $\mathbf{C}_\Pi$, $\mathbf{C}_{\substack{\Pi \\ \Pi'}}$ and $\mathbf{C}_{\tilde{\Pi}}$ of the asymptotic distribution of* $\text{vec}(\mathbf{\Pi}_{y,T})$, $\text{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ *and* $\text{vec}(\mathbf{\Pi}_{\tilde{y},T})$ *are given by*

$$\mathbf{C}_\Pi = (\mathbf{\Pi}_y^* \otimes \mathbf{U}) + (\mathbf{U}^* \otimes \mathbf{\Pi}_y) \tag{3}$$

$$\mathbf{C}_{\substack{\Pi \\ \Pi'}} = \begin{pmatrix} \mathbf{\Pi}_y^* \otimes \mathbf{U} & \mathbf{\Pi}_y^* \otimes \mathbf{U}'' \\ \mathbf{\Pi}_y^* \otimes \mathbf{U}''^H & \mathbf{\Pi}_y^* \otimes \mathbf{U}' \end{pmatrix} + \begin{pmatrix} \mathbf{U}^* \otimes \mathbf{\Pi}_y & \mathbf{U}''^* \otimes \mathbf{\Pi}_y \\ \mathbf{U}''^T \otimes \mathbf{\Pi}_y & \mathbf{U}'^* \otimes \mathbf{\Pi}_y \end{pmatrix} \tag{4}$$

$$\mathbf{C}_{\tilde{\Pi}} = (\mathbf{I} + \mathbf{K}(\mathbf{J} \otimes \mathbf{J}))\left((\mathbf{\Pi}_{\tilde{y}}^* \otimes \tilde{\mathbf{U}}) + (\tilde{\mathbf{U}}^* \otimes \mathbf{\Pi}_{\tilde{y}})\right) \tag{5}$$

with $\mathbf{U} \stackrel{\text{def}}{=} \sigma^2 \mathbf{R}_s^{\#} \mathbf{R}_y \mathbf{R}_s^{\#}$, $\mathbf{U}' \stackrel{\text{def}}{=} \sigma^2 \mathbf{R}_s'^{*\#} \mathbf{R}_y^* \mathbf{R}_s'^{\#}$, $\mathbf{U}'' \stackrel{\text{def}}{=} \sigma^2 \mathbf{R}_s^{\#} \mathbf{R}'_y \mathbf{R}_s'^{\#}$ *and* $\tilde{\mathbf{U}} \stackrel{\text{def}}{=} \sigma^2 \mathbf{R}_{\tilde{s}}^{\#} \mathbf{R}_{\tilde{y}} \mathbf{R}_{\tilde{s}}^{\#}$, *and where $\mathbf{K}$ is the vec-permutation matrix which transforms* $\text{vec}(.)$ *to* $\text{vec}(.^T)$ *for any square matrix and* $\mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}$.

We note that the previous expressions of $\mathbf{C}_\Pi$, $\mathbf{C}_{\substack{\Pi \\ \Pi'}}$ and $\mathbf{C}_{\tilde{\Pi}}$, do not depend on the temporal correlation and the fourth-order moments of $\mathbf{x}_t$. Furthermore, $\mathbf{C}_\Pi$ does not depend on $\mathbf{R}'_y$. Consequently we have proved the following:

Theorem 1 *The asymptotic performance given by an arbitrary consistent subspace-based algorithm built from $\mathbf{\Pi}_{y,T}$, $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ or $\mathbf{\Pi}_{\tilde{y},T}$ depends on the distribution of the time series $\mathbf{x}_t$ through the second-order moments of $\mathbf{x}_t$ only. Furthermore, for subspace-based algorithms built from $\mathbf{\Pi}_{y,T}$, this asymptotic performance depends only on the first covariance matrix $\mathbf{R}_x$.*

3.2 Relation to the Cramer-Rao bound in the Gaussian case

To evaluate the efficiency of the subspace-based AMV estimators previously introduced, we consider the particular case where the signals $\mathbf{x}_t$ are independently Gaussian distributed. The following main contribution of this paper is proved in the Appendix.

Theorem 2 *When the signals are independently Gaussian distributed, the AMV bound* (2) *associated with the statistics* $\text{vec}(\mathbf{\Pi}_{y,T})$, *[resp.* $\text{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ *and* $\text{vec}(\mathbf{\Pi}_{\tilde{y},T})$*] are equal to the statistical CRB associated with the circular [resp. noncircular] Gaussian distribution.*

$$\mathbf{C}_\Theta^{\text{AMV}(\Pi)} = \mathbf{CRB}_\Theta^{\text{CG}} = \frac{\sigma^2}{2}\left\{\Re\left[\frac{d^H \text{vec}\mathbf{A}}{d\Theta}(\mathbf{H}^T \otimes \mathbf{\Pi}_y)\frac{d\text{vec}\mathbf{A}}{d\Theta}\right]\right\}^{-1} \tag{6}$$

$$\mathbf{C}_\Theta^{\text{AMV}(\Pi,\Pi')} = \mathbf{CRB}_\Theta^{\text{NCG}} = \frac{\sigma^2}{2}\left\{\Re\left[\frac{d^H \text{vec}\mathbf{A}}{d\Theta}(\mathbf{H}_1^T \otimes \mathbf{\Pi}_y)\frac{d\text{vec}\mathbf{A}}{d\Theta}\right]\right\}^{-1} \tag{7}$$

$$\mathbf{C}_\Theta^{\text{AMV}(\tilde{\Pi})} = \mathbf{CRB}_\Theta^{\text{NCG}}. \tag{8}$$

with $\mathbf{H} \stackrel{\text{def}}{=} \mathbf{R}_x \mathbf{A}^H \mathbf{R}_y^{-1} \mathbf{A} \mathbf{R}_x$ *and* $\mathbf{H}_1 \stackrel{\text{def}}{=} [\mathbf{R}_x \mathbf{A}^H, \mathbf{R}'_x \mathbf{A}^T] \mathbf{R}_{\tilde{y}}^{-1} \begin{bmatrix} \mathbf{A}\mathbf{R}_x \\ \mathbf{A}^* \mathbf{R}_x'^* \end{bmatrix}$.

Remark 1: Note that for DOA parameterizations,

$$\mathbf{A}(\Theta) = (\mathbf{a}(\theta_1, ..., \theta_{K_1}), ..., \mathbf{a}(\theta_{(K_2-1)K_1+1}, ..., \theta_{K_2 K_1}))$$

for K_1 parameters per source with K_2 sources ($K = K_1 K_2$) and consequently, relations (6) and (7) can be written in the following forms:

$$\mathbf{CRB}_\Theta^{\text{CG}} = \frac{\sigma^2}{2}\left\{\Re\left[\mathbf{D}^H \mathbf{\Pi}_y \mathbf{D} \odot \left(\mathbf{H}^T \otimes \mathbf{1}\right)\right]\right\}^{-1}$$

$$\mathbf{CRB}_\Theta^{\text{NCG}} = \frac{\sigma^2}{2}\left\{\Re\left[\mathbf{D}^H \mathbf{\Pi}_y \mathbf{D} \odot \left(\mathbf{H}_1^T \otimes \mathbf{1}\right)\right]\right\}^{-1}$$

where $\mathbf{1}$ is a $K_1 \times K_1$ matrix of 1 and $\mathbf{D} \stackrel{\text{def}}{=} \left[\frac{\partial \mathbf{a}(\theta_1, .., \theta_{K_1})}{\partial \theta_1}, .., \frac{\partial \mathbf{a}(\theta_{(K_2-1)K_1+1}, .., \theta_{K_2 K_1})}{\partial \theta_{K_2 K_1}}\right]$.

Remark 2: Using the asymptotic equivalence between optimal subspace fitting and optimal subspace matching estimates proved in [4], this result states that all these algorithms give efficient estimates of Θ in the independent Gaussian case. Note that this result has been indirectly proved in [5] for the subspace fitting approach introduced by Viberg in the context of DOA of Gaussian circular sources by comparaison to the stochastic maximum likelihood estimator which is asymptotically efficient in the Gaussian case.

Remark 3: Because the statistic $\mathbf{\Pi}_{y,T}$ is a function of $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$, we have $\mathbf{C}_\Theta^{\mathrm{AMV}(\Pi,\Pi')} \leq \mathbf{C}_\Theta^{\mathrm{AMV}(\Pi)}$ and consequently $\mathbf{CRB}_\Theta^{\mathrm{NCG}} \leq \mathbf{CRB}_\Theta^{\mathrm{CG}}$ for independent Gaussian signals of same first spatial covariance matrices $\mathbf{R}_x$.

4. ILLUSTRATIVE EXAMPLE

Consider now the specific case of blind identification of noisy SISO FIR channels of order $M-1$, $y_t = \sum_{k=0}^{M-1} h_k x_{t-k} + n_t$, whose input x_t is a sequence of independent BPSK symbols $\{-1,+1\}$ with equal probabilities. By stacking M samples of the received signal, we obtain the vector:

$$\mathbf{y}_t \stackrel{\mathrm{def}}{=} (y_t, y_{t-1}, ..., y_{t-M+1})^T = \mathbf{A}(\Theta)\mathbf{x}_t + \mathbf{n}_t$$

with $\mathbf{x}_t \stackrel{\mathrm{def}}{=} (x_t, x_{t-1}, ..., x_{t-2M+2})^T$ and where $\mathbf{A}(\Theta)$ is the following $M \times (2M-1)$ filtering matrix

$$\mathbf{A}(\Theta) = \begin{pmatrix} h_0 & \cdots & \cdots & h_{M-1} & & & \\ & \ddots & & & \ddots & & \\ & & h_0 & \cdots & \cdots & h_{M-1} \end{pmatrix},$$

with $h_0 = 1$. In this case $\Theta = [\Re(h_1), ..\Re(h_{M-1}), Im(h_1), .., Im(h_{M-1})]^T \in \mathbb{R}^K$, with $K = 2(M-1)$. Naturally, Θ is not identifiable from the second-order information $\mathbf{R}_y$ alone and consequently not from $\mathbf{\Pi}_y$ alone either. But with $(\mathbf{R}_y, \mathbf{R}'_y)$, it becomes identifiable [8] if and only if the polynomial $h(z) = \sum_{k=0}^{M-1} h_k z^k$ has no real zero and no conjugated zeros. In this case, subspace-based algorithms can be considered from the extended observation $\tilde{\mathbf{y}}_t$

$$\tilde{\mathbf{y}}_t = \begin{bmatrix} \mathbf{A}(\Theta) \\ \mathbf{A}^*(\Theta) \end{bmatrix} \mathbf{x}_t + \tilde{\mathbf{n}}_t$$

and the AMV bound (6) applies when the a priori knowledge of the independence of the symbols is not taken into account, where here $\mathbf{A}(\Theta)$ and $\mathbf{R}_y$ are replaced by $\begin{bmatrix} \mathbf{A}(\Theta) \\ \mathbf{A}^*(\Theta) \end{bmatrix}$ and $\mathbf{R}_{\tilde{y}} = \begin{bmatrix} \mathbf{A}(\Theta) \\ \mathbf{A}^*(\Theta) \end{bmatrix} \begin{bmatrix} \mathbf{A}^H(\Theta) & \mathbf{A}^T(\Theta) \end{bmatrix} + \sigma^2 \mathbf{I}_{2M}$.

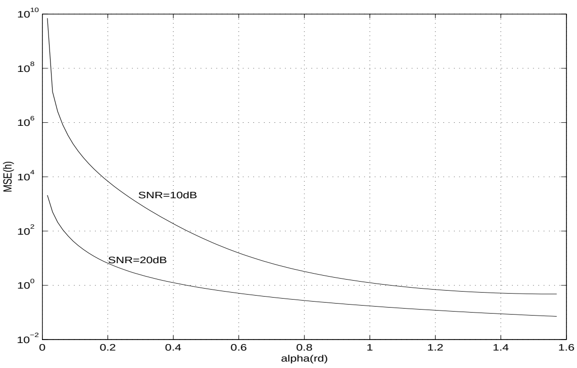

Fig.1 Normalized MSE($\mathbf{h}$)=Tr($\mathbf{C}_\Theta^{\mathrm{AMV}}$) as a function of the phase α of the root z_1 for SNR= 10dB and 20dB.

Fig.1 exhibits the normalized MSE($\mathbf{h}$)=Tr($\mathbf{C}_\Theta^{\mathrm{AMV}}$) which is a lower bound on the MSE of any unbiased estimate based on the projector $\mathbf{\Pi}_{\tilde{y},T}$. $h(z) = (1 - z_1^{-1} z)(1 - z_2^{-1} z)$ with $z_1 = 0.8e^{i\alpha}$ and $z_2 = 1.25e^{i\pi/4}$ where α varies from 0 to $\pi/2$. We note that the MSE increases dramatically when the zero z_1 approaches the real axis for which Θ becomes nonidentifiable. This behavior is explained by (6) for which $\begin{bmatrix} \mathbf{A}(\Theta) \\ \mathbf{A}^*(\Theta) \end{bmatrix}$ becomes rank deficient and consequently, the associated matrix $\mathbf{H}$ becomes rank deficient as well.

A. APPENDIX: PROOF OF THEOREM 2

We separately consider the three statistics where we will make relatively frequent use of the following identities:

$$\begin{aligned} \mathrm{vec}(\mathbf{ABC}) &= (\mathbf{C}^T \otimes \mathbf{A})\mathrm{vec}(\mathbf{B}) & (9) \\ \mathrm{Tr}(\mathbf{ABCD}) &= \mathrm{vec}^T(\mathbf{A}^T)(\mathbf{D}^T \otimes \mathbf{B})\mathrm{vec}(\mathbf{C}) & (10) \end{aligned}$$

Projector vec($\mathbf{\Pi}_{y,T}$):

Because Nullspace$(\mathbf{R}_s^\#) = \mathrm{Span}(\mathbf{\Pi}_y)$, we have $\mathbf{U}\mathbf{\Pi}_y = \mathbf{O}$. This implies the two relations

$$\begin{aligned} (\mathbf{\Pi}_y^* \otimes \mathbf{U})(\mathbf{U}^* \otimes \mathbf{\Pi}_y)^H &= \mathbf{\Pi}_y^* \mathbf{U}^T \otimes \mathbf{U}\mathbf{\Pi}_y = \mathbf{O} \\ (\mathbf{\Pi}_y^* \otimes \mathbf{U})^H(\mathbf{U}^* \otimes \mathbf{\Pi}_y) &= \mathbf{\Pi}_y^* \mathbf{U}^* \otimes \mathbf{U}\mathbf{\Pi}_y = \mathbf{O}, \end{aligned}$$

which, thanks to [11, Th.5.17], enables one to write the Moore-Penrose inverse of $\mathbf{C}_\Pi$ given by (3) in the form:

$$\begin{aligned} \mathbf{C}_\Pi^\# &= (\mathbf{\Pi}_y^* \otimes \mathbf{U})^\# + (\mathbf{U}^* \otimes \mathbf{\Pi}_y)^\# \\ &= (\mathbf{\Pi}_y^{\#*} \otimes \mathbf{U}^\#) \\ &+ (\mathbf{U}^{\#*} \otimes \mathbf{\Pi}_y^\#) = (\mathbf{\Pi}_y^* \otimes \mathbf{U}^\#) + (\mathbf{U}^{\#*} \otimes \mathbf{\Pi}_y) \\ &= \frac{1}{\sigma^2}\left((\mathbf{\Pi}_y^* \otimes \mathbf{AHA}^H) + (\mathbf{A}^*\mathbf{H}^*\mathbf{A}^T \otimes \mathbf{\Pi}_y)\right) \quad (11) \end{aligned}$$

where the second equality is by [12, Th.5 (xvii), p.33] and the last equality is deduced from $\mathbf{U}^\# = \frac{1}{\sigma^2}\mathbf{R}_s\mathbf{R}_y^{-1}\mathbf{R}_s = \frac{1}{\sigma^2}\mathbf{A}\mathbf{R}_x\mathbf{A}^H\mathbf{R}_y^{-1}\mathbf{A}\mathbf{R}_x\mathbf{A}^H = \frac{1}{\sigma^2}\mathbf{AHA}^H$ with $\mathbf{H} \stackrel{\mathrm{def}}{=} \mathbf{R}_x\mathbf{A}^H\mathbf{R}_y^{-1}\mathbf{A}\mathbf{R}_x$, thanks to [11, Th.5.6 and Th.5.7] because the Hermitian matrices $\mathbf{R}_s$ and $\mathbf{R}_y$ have a common basis of orthonormal eigenvectors. So, from Theorem 1

$$\begin{aligned} \left[\left(\mathbf{C}_\Theta^{\mathrm{AMV}(\Pi)}\right)^{-1}\right]_{k,l} &= \frac{1}{\sigma^2}\frac{\partial \mathrm{vec}^T(\mathbf{\Pi}_y^T)}{\partial \theta_k}\left((\mathbf{\Pi}_y^T \otimes \mathbf{AHA}^H)\right. \\ &+ \left.((\mathbf{AHA}^H)^T \otimes \mathbf{\Pi}_y)\right)\frac{\partial \mathrm{vec}(\mathbf{\Pi}_y)}{\partial \theta_l} \\ &= \frac{1}{\sigma^2}\mathrm{Tr}\left(\frac{\partial \mathbf{\Pi}_y}{\partial \theta_k}\mathbf{AHA}^H\frac{\partial \mathbf{\Pi}_y}{\partial \theta_l}\mathbf{\Pi}_y\right. \\ &+ \left.\frac{\partial \mathbf{\Pi}_y}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{\Pi}_y}{\partial \theta_l}\mathbf{AHA}^H\right) \\ &= \frac{2}{\sigma^2}\Re\left[\mathrm{Tr}\left(\mathbf{A}^H\frac{\partial \mathbf{\Pi}_y}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{\Pi}_y}{\partial \theta_l}\mathbf{AH}\right)\right], \end{aligned}$$

where we have used identity (10) in the second equality. Then $\mathbf{\Pi}_y\mathbf{A} = \mathbf{O}$ implying

$$\frac{\partial \mathbf{\Pi}_y}{\partial \theta_i}\mathbf{A} + \mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial \theta_i} = \mathbf{O}, \quad i = k, l, \qquad (12)$$

we have

$$\left[\left(\mathbf{C}_\Theta^{\mathrm{AMV}(\Pi)}\right)^{-1}\right]_{k,l} = \frac{2}{\sigma^2}\Re\left[\mathrm{Tr}\left(\frac{\partial \mathbf{A}^H}{\partial\theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial\theta_l}\mathbf{H}\right)\right]. \quad (13)$$

This proves the second part of (6) thanks to (10). To obtain the associated CRB, we follow the proof given in [13] where the CRB for the DOA parameter is directly derived from the Slepian-Bangs formula. All the steps of this proof apply where [13, rel. (16)] is replaced by $\frac{\partial \mathbf{R}_y}{\partial\theta_k} = \frac{\partial \mathbf{A}}{\partial\theta_k}\mathbf{R}_x\mathbf{A}^H + \mathbf{A}\mathbf{R}_x\frac{\partial \mathbf{A}^H}{\partial\theta_k}$ and where [13, rel. (18)] becomes $\mathbf{Z}_k = \mathbf{R}_y^{-1/2}\mathbf{A}\mathbf{R}_x\frac{\partial \mathbf{A}^H}{\partial\theta_k}\mathbf{R}_y^{-1/2}$. Consequently [13, rel. (30)] becomes, (6) thanks to [13, rel. (31)] and (10).

Projector $\mathrm{vec}(\mathbf{\Pi}_{y,T},\mathbf{\Pi}'_{y,T})$:

As for the statistic $\mathrm{vec}(\mathbf{\Pi}_{y,T})$, we have $\mathbf{U}\mathbf{\Pi}_y = \mathbf{U}'\mathbf{\Pi}_y = \mathbf{U}''\mathbf{\Pi}_y = \mathbf{O}$, which implies after straightforward algebraic manipulations, the two relations

$$\begin{pmatrix} \mathbf{\Pi}_y^*\otimes\mathbf{U} & \mathbf{\Pi}_y^*\otimes\mathbf{U}'' \\ \mathbf{\Pi}_y^*\otimes\mathbf{U}''^H & \mathbf{\Pi}_y^*\otimes\mathbf{U}' \end{pmatrix} \begin{pmatrix} \mathbf{U}^*\otimes\mathbf{\Pi}_y & \mathbf{U}''^*\otimes\mathbf{\Pi}_y \\ \mathbf{U}''^T\otimes\mathbf{\Pi}_y & \mathbf{U}'^*\otimes\mathbf{\Pi}_y \end{pmatrix}^H = \mathbf{O} \quad (14)$$

$$\begin{pmatrix} \mathbf{\Pi}_y^*\otimes\mathbf{U} & \mathbf{\Pi}_y^*\otimes\mathbf{U}'' \\ \mathbf{\Pi}_y^*\otimes\mathbf{U}''^H & \mathbf{\Pi}_y^*\otimes\mathbf{U}' \end{pmatrix}^H \begin{pmatrix} \mathbf{U}^*\otimes\mathbf{\Pi}_y & \mathbf{U}''^*\otimes\mathbf{\Pi}_y \\ \mathbf{U}''^T\otimes\mathbf{\Pi}_y & \mathbf{U}'^*\otimes\mathbf{\Pi}_y \end{pmatrix} = \mathbf{O}. \quad (15)$$

This enables one to write, thanks to [11, Th.5.17], the Moore-Penrose inverse of $\mathbf{C}_{\substack{\Pi\\ \Pi'}}$ given by (4) in the form:

$$\begin{aligned}
\mathbf{C}^\#_{\substack{\Pi\\ \Pi'}} &= \begin{pmatrix} \mathbf{\Pi}_y^*\otimes\mathbf{U} & \mathbf{\Pi}_y^*\otimes\mathbf{U}'' \\ \mathbf{\Pi}_y^*\otimes\mathbf{U}''^H & \mathbf{\Pi}_y^*\otimes\mathbf{U}' \end{pmatrix}^\# + \begin{pmatrix} \mathbf{U}^*\otimes\mathbf{\Pi}_y & \mathbf{U}''^*\otimes\mathbf{\Pi}_y \\ \mathbf{U}''^T\otimes\mathbf{\Pi}_y & \mathbf{U}'^*\otimes\mathbf{\Pi}_y \end{pmatrix}^\# \\
&= \left(\begin{pmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K} \end{pmatrix}\left(\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^H & \mathbf{U}' \end{pmatrix}\otimes\mathbf{\Pi}_y^*\right)\begin{pmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K} \end{pmatrix}\right)^\# + \left(\begin{pmatrix} \mathbf{U}^* & \mathbf{U}''^* \\ \mathbf{U}''^T & \mathbf{U}'^* \end{pmatrix}\otimes\mathbf{\Pi}_y\right)^\# \\
&= \left(\begin{pmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K} \end{pmatrix}\left(\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^H & \mathbf{U}' \end{pmatrix}^\#\otimes\mathbf{\Pi}_y^*\right)\begin{pmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K} \end{pmatrix}\right) + \left(\begin{pmatrix} \mathbf{U}^* & \mathbf{U}''^* \\ \mathbf{U}''^T & \mathbf{U}'^* \end{pmatrix}^\#\otimes\mathbf{\Pi}_y\right), \quad (16)
\end{aligned}$$

where we have used the identity $\mathbf{A}\otimes\mathbf{B} = \mathbf{K}(\mathbf{B}\otimes\mathbf{A})\mathbf{K}$ [12, Th.(4), p.47] in the second equality, and [11, Th.5.8] and [12, Th.5 (xvii), p.33] in the third equality. Noting that

$$\begin{pmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K} \end{pmatrix}\frac{\partial\mathrm{vec}(\mathbf{\Pi}_y,\mathbf{\Pi}_y)}{\partial\theta_i} = \begin{pmatrix} \mathbf{K}\frac{\partial\mathrm{vec}(\mathbf{\Pi}_y)}{\partial\theta_i} \\ \mathbf{K}\frac{\partial\mathrm{vec}(\mathbf{\Pi}_y)}{\partial\theta_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial\mathrm{vec}(\mathbf{\Pi}_y^*)}{\partial\theta_i} \\ \frac{\partial\mathrm{vec}(\mathbf{\Pi}_y^*)}{\partial\theta_i} \end{pmatrix}, \quad i = k,l,$$

we have from (16)

$$\begin{aligned}
\left[\left(\mathbf{C}_\Theta^{\mathrm{AMV}(\Pi,\Pi')}\right)^{-1}\right]_{k,l} &= 2\Re\left[\frac{\partial\mathrm{vec}^T}{\partial\theta_k}\begin{pmatrix} \mathbf{\Pi}_y \\ \mathbf{\Pi}_y \end{pmatrix}^T \left(\left(\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^H & \mathbf{U}' \end{pmatrix}^\#\right)^T\otimes\mathbf{\Pi}_y\right)\frac{\partial\mathrm{vec}(\mathbf{\Pi}_y,\mathbf{\Pi}_y)}{\partial\theta_l}\right] \\
&= 2\Re\left[\mathrm{Tr}\left(\begin{pmatrix} \frac{\partial\mathbf{\Pi}_y}{\partial\theta_k} \\ \frac{\partial\mathbf{\Pi}_y}{\partial\theta_k} \end{pmatrix}\mathbf{\Pi}_y \begin{pmatrix} \frac{\partial\mathbf{\Pi}_y}{\partial\theta_l} & \frac{\partial\mathbf{\Pi}_y}{\partial\theta_l} \end{pmatrix}\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^H & \mathbf{U}' \end{pmatrix}^\#\right)\right]
\end{aligned}$$

where identity (10) is used in the second equality. Then from the definition of the matrices $\mathbf{U}$, $\mathbf{U}'$ and $\mathbf{U}''$ given in Lemma 1, we have

$$\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^H & \mathbf{U}' \end{pmatrix} = \sigma^2\begin{pmatrix} \mathbf{R}_s^\# & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{*\#} \end{pmatrix}\mathbf{R}_{\tilde{y}}\begin{pmatrix} \mathbf{R}_s^\# & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^\# \end{pmatrix},$$

whose Moore-Penrose inverse is given in the following thanks to [11, Th.5.14] and extension of [11, Th.5.6 and Th.5.7] to the singular value decompositions of the matrices $\begin{pmatrix} \mathbf{R}_s^\# & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{*\#} \end{pmatrix}$, $\mathbf{R}_{\tilde{y}}$ and $\begin{pmatrix} \mathbf{R}_s^\# & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^\# \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^H & \mathbf{U}' \end{pmatrix}^\# &= \frac{1}{\sigma^2}\begin{pmatrix} \mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s' \end{pmatrix}\mathbf{R}_{\tilde{y}}^{-1}\begin{pmatrix} \mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^* \end{pmatrix} \\
&= \frac{1}{\sigma^2}\begin{pmatrix} \mathbf{A}\mathbf{R}_x\mathbf{A}^H & \mathbf{O} \\ \mathbf{O} & \mathbf{A}\mathbf{R}_x'\mathbf{A}^T \end{pmatrix}\mathbf{R}_{\tilde{y}}^{-1}\begin{pmatrix} \mathbf{A}\mathbf{R}_x\mathbf{A}^H & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^*\mathbf{R}_x'^*\mathbf{A}^H \end{pmatrix} \\
&= \frac{1}{\sigma^2}\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix}\begin{pmatrix} \mathbf{R}_x\mathbf{A}^H & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_x'\mathbf{A}^T \end{pmatrix}\mathbf{R}_{\tilde{y}}^{-1}\begin{pmatrix} \mathbf{A}\mathbf{R}_x & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^*\mathbf{R}_x'^* \end{pmatrix}\begin{pmatrix} \mathbf{A}^H & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^H \end{pmatrix} \\
&= \frac{1}{\sigma^2}\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix}\tilde{\mathscr{H}}\begin{pmatrix} \mathbf{A}^H & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^H \end{pmatrix},
\end{aligned}$$

with $\tilde{\mathscr{H}} \stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathbf{R}_x\mathbf{A}^H & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_x'\mathbf{A}^T \end{pmatrix}\mathbf{R}_{\tilde{y}}^{-1}\begin{pmatrix} \mathbf{A}\mathbf{R}_x & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^*\mathbf{R}_x'^* \end{pmatrix}$.
Consequently,

$$\begin{aligned}
\left[\left(\mathbf{C}_\Theta^{\mathrm{AMV}(\Pi,\Pi')}\right)^{-1}\right]_{k,l} &= \frac{2}{\sigma^2}\Re\left[\mathrm{Tr}\left(\begin{pmatrix} \frac{\partial\mathbf{\Pi}_y}{\partial\theta_k} \\ \frac{\partial\mathbf{\Pi}_y}{\partial\theta_k} \end{pmatrix}\mathbf{\Pi}_y\begin{pmatrix} \frac{\partial\mathbf{\Pi}_y}{\partial\theta_l} & \frac{\partial\mathbf{\Pi}_y}{\partial\theta_l} \end{pmatrix}\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix}\tilde{\mathscr{H}}\begin{pmatrix} \mathbf{A}^H & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^H \end{pmatrix}\right)\right] \\
&= \frac{2}{\sigma^2}\Re\left[\mathrm{Tr}\left(\begin{pmatrix} \mathbf{A}^H\frac{\partial\mathbf{\Pi}_y}{\partial\theta_k} \\ \mathbf{A}^H\frac{\partial\mathbf{\Pi}_y}{\partial\theta_k} \end{pmatrix}\mathbf{\Pi}_y\begin{pmatrix} \frac{\partial\mathbf{\Pi}_y}{\partial\theta_l}\mathbf{A} & \frac{\partial\mathbf{\Pi}_y}{\partial\theta_l}\mathbf{A} \end{pmatrix}\tilde{\mathscr{H}}\right)\right].
\end{aligned}$$

Applying identity (12), we obtain:

$$\left[\left(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\Pi,\Pi')}\right)^{-1}\right]_{k,l} = \frac{2}{\sigma^2} \Re\left[\mathrm{Tr}\left(\left(\begin{array}{cc} \frac{\partial \mathbf{A}^H}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial \theta_l} & \frac{\partial \mathbf{A}^H}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial \theta_l} \\ \frac{\partial \mathbf{A}^H}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial \theta_l} & \frac{\partial \mathbf{A}^H}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial \theta_l}\end{array}\right)\mathscr{H}\right)\right]$$

which gives after straightforward algebraic manipulations

$$\left[\left(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})}\right)^{-1}\right]_{k,l} = \frac{2}{\sigma^2} \Re\left[\mathrm{Tr}\left(\frac{\partial \mathbf{A}^H}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial \theta_l}[\mathbf{R}_x\mathbf{A}^H,\mathbf{R}'_x\mathbf{A}^T]\mathbf{R}_{\tilde{y}}^{-1}\left[\begin{array}{c}\mathbf{A}\mathbf{R}_x \\ \mathbf{A}^*\mathbf{R}'^*_x\end{array}\right]\right)\right].$$

This proves the second part of (7) thanks to (10). To obtain the associated CRB, we follow the proof given in [14] for the DOA parameters of noncircular sources where

$$\mathbf{Z}_k = \mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{R}_{\tilde{y}}\frac{\partial \tilde{\mathbf{A}}^H}{\partial \theta_k}\mathbf{R}_{\tilde{y}}^{-1/2}. \qquad \blacksquare$$

Projector $\mathrm{vec}(\mathbf{\Pi}_{\tilde{y},T})$:

To prove Theorem 2 for this statistic, we first must simplify the expression of $\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})}$. Because $\mathbf{L} \stackrel{\mathrm{def}}{=} \mathbf{I}+\mathbf{K}(\mathbf{J}\otimes\mathbf{J})$ of (5) satisfies $\mathbf{L}^2 = 2\mathbf{L}$, the Hermitian matrix $\mathbf{C}_{\tilde{\Pi}}$ becomes: $\mathbf{C}_{\tilde{\Pi}} = \frac{1}{2}\mathbf{LCL}$ with $\mathbf{C} \stackrel{\mathrm{def}}{=} (\mathbf{\Pi}_{\tilde{y}}^*\otimes\tilde{\mathbf{U}})+(\tilde{\mathbf{U}}^*\otimes\mathbf{\Pi}_{\tilde{y}})$ and a simpler expression of the AMV bound can be obtained from the following minimization problem:

$$\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})} = \min_{\mathbf{D}\mathscr{S}=\mathbf{I}_K}\mathbf{D}\mathbf{C}_{\tilde{\Pi}}\mathbf{D}^H = \frac{1}{2}\min_{\mathbf{D}\mathscr{S}=\mathbf{I}_K}\mathbf{DLCLD}^H.$$

Checking that $\mathbf{L}\mathscr{S} = (\mathbf{I}+\mathbf{K}(\mathbf{J}\otimes\mathbf{J}))\frac{d\mathrm{vec}(\mathbf{\Pi}_{\tilde{y}})}{d\Theta} = \mathscr{S} + \mathbf{K}\mathrm{vec}\left(\mathbf{J}\frac{d\mathbf{\Pi}_{\tilde{y}}}{d\Theta}\mathbf{J}\right) = \mathscr{S} + \mathbf{K}\mathrm{vec}\left(\frac{d\mathbf{\Pi}_{\tilde{y}}}{d\Theta}^T\right) = 2\mathscr{S}$ thanks to identity (9) for the second equality and the property $\mathbf{J}\mathbf{\Pi}_{\tilde{y}}\mathbf{J} = \mathbf{\Pi}_{\tilde{y}}^T$ [15] for the third equality; the constraints $\mathbf{D}\mathscr{S} = \mathbf{I}$ and $\mathbf{DL}\mathscr{S} = 2\mathbf{I}$ are equivalent. Consequently the previous minimization is tantamount to

$$\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})} = 2\min_{\frac{\mathbf{DL}}{2}\mathscr{S}=\mathbf{I}_K}\left(\frac{\mathbf{DL}}{2}\right)\mathbf{C}\left(\frac{\mathbf{DL}}{2}\right)^H.$$

Because $\mathbf{C}$ is structured similarly as $\mathbf{C}_{\Pi}$ (see (3)), $\mathrm{Span}(\mathscr{S}) \subset \mathrm{Span}(\mathbf{C})$. Consequently, the proof of Theorem 1 given in [10] applies and $\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})} = 2\left(\mathscr{S}^H\mathbf{C}^{\#}\mathscr{S}\right)^{-1}$.

Noting that $\mathbf{C} = (\mathbf{\Pi}_{\tilde{y}}^*\otimes\tilde{\mathbf{U}})+(\tilde{\mathbf{U}}^*\otimes\mathbf{\Pi}_{\tilde{y}})$ is structured similarly to $\mathbf{C}_{\Pi}$, all the steps of the proof given for the statistic $\mathrm{vec}(\mathbf{\Pi}_{y,T})$ extend up to equality (13) by replacing $\mathbf{A}$, $\mathbf{\Pi}_y$ and $\mathbf{H} = \mathbf{R}_x\mathbf{A}^H\mathbf{R}_y^{-1}\mathbf{A}\mathbf{R}_x$, by $\tilde{\mathbf{A}}$, $\mathbf{\Pi}_{\tilde{y}} = \left(\begin{array}{cc}\mathbf{\Pi}_y & \mathbf{O} \\ \mathbf{O} & \mathbf{\Pi}_y^*\end{array}\right)$ (from [15]) and $\tilde{\mathbf{H}} \stackrel{\mathrm{def}}{=} \mathbf{R}_{\tilde{x}}\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}}\mathbf{R}_{\tilde{x}}$ respectively, and consequently

$$\left[\left(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})}\right)^{-1}\right]_{k,l} = \frac{1}{2}\frac{2}{\sigma^2}\Re\left[\mathrm{Tr}\left(\frac{\partial \tilde{\mathbf{A}}^H}{\partial \theta_k}\mathbf{\Pi}_{\tilde{y}}\frac{\partial \tilde{\mathbf{A}}}{\partial \theta_l}\tilde{\mathbf{H}}\right)\right].$$

Because all the matrices involved in $\tilde{\mathbf{H}}$ are structured in the form $\left(\begin{array}{cc}(\square) & (\times) \\ (\times)^* & (\square)^*\end{array}\right)$, $\tilde{\mathbf{H}}$ is structured in the same form as well, i.e., $\tilde{\mathbf{H}} = \left(\begin{array}{cc}\mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2^* & \mathbf{H}_1^*\end{array}\right)$ with $\mathbf{H}_1 = [\mathbf{R}_x\mathbf{A}^H,\mathbf{R}'_x\mathbf{A}^T]\mathbf{R}_{\tilde{y}}^{-1}\left[\begin{array}{c}\mathbf{A}\mathbf{R}_x \\ \mathbf{A}^*\mathbf{R}'^*_x\end{array}\right]$. Then

$$\frac{\partial \tilde{\mathbf{A}}^H}{\partial \theta_k}\mathbf{\Pi}_{\tilde{y}}\frac{\partial \tilde{\mathbf{A}}}{\partial \theta_l}\tilde{\mathbf{H}} = \left(\begin{array}{cc}\frac{\partial \mathbf{A}^H}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial \theta_l}\mathbf{H}_1 & (\times) \\ (\times)^* & \frac{\partial \mathbf{A}^T}{\partial \theta_k}\mathbf{\Pi}_y^*\frac{\partial \mathbf{A}^*}{\partial \theta_l}\mathbf{H}_1^*\end{array}\right)$$

and

$$\left[\left(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})}\right)^{-1}\right]_{k,l} = \frac{2}{\sigma^2}\Re\mathrm{Tr}\left(\frac{\partial \mathbf{A}^H}{\partial \theta_k}\mathbf{\Pi}_y\frac{\partial \mathbf{A}}{\partial \theta_l}\mathbf{H}_1\right),$$

which using (10), proves (8) thanks to (7). $\blacksquare$

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