WLS QUASI-EQUIRIPPLE DESIGN OF VARIABLE FRACTIONAL DELAY FIR FILTERS

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ABSTRACT

To design an FIR filter, the Weighted Least Squares (WLS) method is a well-known technique. And it has already been extended to design a variable fractional delay FIR filter. It is reported that the efficiency is better than the method based on Lagrange interpolation. In this paper, the WLS method cooperated with an iterative procedure for obtaining a suitable frequency response weighting function, a quasi-equiripple variable fractional delay filter can be achieved. Simulation results show that after no more than eight iterations we can get rather satisfied results. Besides, the proposed design is superior to the plain WLS (weighting function is constant and fixed) design in peak absolute error by 6.7 dB.

1. INTRODUCTION

Fractional delay filters are the filters which can delay the fractional samples of a discrete-time signal, have many applications in the DSP area, such as sampling rate conversion, speech coding and acoustic system modeling [1]. In a general FIR filter design, only fixed delay filters can be designed. The delay, or phase, of the desired frequency response must be decided before designing the filter. But in practical applications, the filter must be capable of changing the delay continuously and sometimes in real-time. Therefore the filter coefficients must be designed to response or be a function of the fractional delay d. In such design, a more complicated frequency response mathematical expression has to be derived previously. This is used in the WLS method to obtain the optimal coefficients.

In general, to design an equiripple filter, the sophisticated optimization algorithm, such as Remez exchange is used. But to use the algorithm, some strict conditions must be checked first. Besides, not all equiripple problems can be solved with the Remez exchange algorithm. On the contrary, using the WLS method and choosing a suitable weighting function is much straightforward and is easy to understand. However, there is still no analytic method to derive the suitable weighting function for equiripple design. If we relax our requirement of minimax optimality to a near-optimal design, then an iterative technique to adjust the weighting function to achieve a quasi-equiripple result is applicable. The design procedure is to modify the weighting function according to the error of the WLS result in the current iteration and then proceed to the next iteration. In this paper, a complete procedure is presented to design a quasi-equiripple variable fractional delay FIR filter. The result is shown in comparison with the result of the WLS method.

2. VARIABLE FRACTIONAL DELAY FILTERS

A fractional delay filter is to make the output signal as the input signal with only some fractional sample delay. So the ideal frequency response can be given by

$$H_{ideal}(e^{jw}) = e^{-jwD} \tag{1}$$

where D is the delay of the filter. D can be split into the integer and fractional part, that is D=Int(D)+d and d is the fractional delay. Now consider the filter's impulse response, [h_0 h_1 … h_N], which corresponds to certain value of d, has the frequency response:

$$H\left(e^{jw}\right) = \sum_{n=0}^{N} h_n e^{-jwn} \tag{2}$$

Because of the filter's frequency response is changing when applying the different delay d, the filter's coefficients should be a function of d. Consider each coefficient is designed as the Kth-order polynomial of d [2][3]. We have

$$h_n(d) = \sum_{k=0}^{K} h_{nk} d^k \tag{3}$$

The overall frequency response is thus

$$H\left(e^{jw}, d\right) = \sum_{k=0}^{K}\sum_{n=0}^{N} h_{nk} d^k e^{-jwn} = \mathbf{Z}^T(e^{jw})\mathbf{H}_M\mathbf{D}(d) \tag{4}$$

where T denotes transposition, and

$$\mathbf{Z}(e^{jw}) = \begin{bmatrix}1 & e^{-jw} & \cdots & e^{-jwN}\end{bmatrix}^T \tag{5}$$

$$\mathbf{D}(d) = \begin{bmatrix}1 & d & \cdots & d^K\end{bmatrix}^T \tag{6}$$

$$\mathbf{H}_M = \begin{bmatrix} h_{00} & \cdots & h_{0K} \\ \vdots & \ddots & \vdots \\ h_{N0} & \cdots & h_{NK} \end{bmatrix} \tag{7}$$

There is another form that is much easier to be written in computer codes.

$$H(e^{jw}, d) = \mathbf{H}_V^T \left(\mathbf{D}(d) \otimes \mathbf{Z}(e^{jw})\right) \tag{8}$$

where $\otimes$ denotes Kronecker product [4] and

$$\mathbf{H}_V = [h_{00} \cdots h_{0k} \; h_{10} \cdots h_{1k} \cdots\cdots h_{N0} \cdots h_{Nk}]^T \tag{9}$$

Now, the frequency response representation of a variable fractional delay filter with arbitrary desired fractional delay d is completely formulated. It is expressed in the matrix form, and can be directly used in the computer program.

3. THE WLS METHOD FOR VARIABLE FRACTIONAL DELAY FILTER

After getting the frequency response of a variable fractional delay filter, the cost function must be defined while performing the WLS operations. In the WLS approach, the object is to minimize the cost function. That is to minimize the square error over the desired range. In the designing of the variable fractional delay filter, the desired range of both frequency and delay must be defined.

$$J = \int_{d_1}^{d_2} \int_{w_1}^{w_2} \left|E(w,d)\right|^2 W(w,d)\, \mathrm{d}w \mathrm{d}d \tag{10}$$

where $E(w, d)$ is the error function defined as $E(w, d)$= $H_{ideal}(e^{jw})$-$H(e^{jw}, d)$ and $W(w, d)$ is the weighting function. The frequency range is $[w_1, w_2]$ and the delay range is $[d_1, d_2]$. In order to perform the calculation, the cost function J is defined over the discrete set of frequency F and delay C.

$$J = \sum_{d \in C} \sum_{w \in F} \left|E(w,d)\right|^2 W(w,d) \tag{11}$$

The cost function is the summation of the weighted square error at these discrete points. If there are p points in the delay range and q points in the frequency range, the error function becomes a $p\times q$ matrix and the weighting function is also a $p\times q$ matrix.

Expanding the cost function and the solution of the coefficients that minimize the cost function is obtained easily via the following derivation.

$$\begin{aligned} J &= \sum_{d \in C} \sum_{w \in F} \left|E(w,d)\right|^2 W(w,d) \\ &= \sum_{d \in C} \sum_{w \in F} \left|H(e^{jw},d) - H_{id}(e^{jw},d)\right|^2 W(w,d) \\ &= \mathbf{H}_V^T \mathbf{R} \mathbf{H}_V - 2\mathbf{H}_V^T \mathbf{S} + T \end{aligned} \tag{12}$$

where

$$\mathbf{R} = \sum_{d \in C} \sum_{\omega \in F} \left(\mathbf{D}(d) \otimes \mathbf{Z}(e^{jw})\right)\left(\mathbf{D}(d) \otimes \mathbf{Z}(e^{jw})\right)^H W(\omega,d)$$

Moreover

$$\mathbf{S} = \sum_{d \in C} \sum_{\omega \in F} \mathbf{D}(d) \otimes \mathrm{Re}\left(\mathbf{Z}(e^{jw}) \mathbf{H}_{id}^*(e^{j\omega},d)\right) W(\omega,d)$$

And

$$T = \sum_{d \in C} \sum_{\omega \in F} H_{id}(e^{jw},d) H_{id}^*(e^{jw},d) W(w,d)$$

To find the optimal coefficients that minimize the cost function J, take derivatives with respect to H_v and set it to be zero. We have

$$2\mathbf{R}\mathbf{H}_V - 2\mathbf{S} = 0 \tag{13}$$

It can be solved using matrix inversion

$$\mathbf{H}_V = \mathbf{R}^{-1}\mathbf{S} \tag{14}$$

The computational complexity is mainly depends on the inversion of the matrix **R**, and the numerical problem may be encountered at the matrix inversion. For fast and accurate calculation of $\mathbf{R}^{-1}$, some techniques may be used by analyzing the matrix **R**, or limiting the weighting function to factorize the matrix **R** [2]. But this problem may not be completely solved. It's the price to pay in the using of WLS method.

4. THE WEIGHTING FUNCTION

In the general FIR filter design, a quasi-equiripple filter designed by adapting the weighting function with iteration techniques was demonstrated in [6]. The weighting function and the error function are calculated over a desired range of frequency. The weighting function $W(w)$ used in the kth iteration will be modified for used in the $(k+1)$th iteration. A simple way of the modification is to construct an updating function $\beta(w)$. The next weight function is the multiplication of the current weight function and the updating function.

$$W_{k+1}(w) = W_k(w)\, \beta(w) \tag{15}$$

The simple rule for the updating function is, for two different frequency w_1 and w_2, if the absolute error $|E(w_1)|$ is greater than $|E(w_2)|$, $\beta(w_1)$ should be greater than $\beta(w_2)$. This ensures the error at frequency w_1 decreases at the next iteration.

In Lawson's scheme [5], the updating function is defined as

$$\beta_k(w_n) = \frac{\left|E_k(w_n)\right|}{\sum_i W_k(w_i)\left|E_k(w_i)\right|} \tag{16}$$

Some disadvantages appear in this function. The convergence of Lawson's algorithm is slow. And if the error is zero at some frequency in kth iteration, the weighting function for the next iteration, $(k+1)$th iteration, will be zero at this frequency. And the zero will propagate in the following iterations at this frequency. To avoid this problem, the envelope of the absolute error is used to construct the updating function [6]. It successfully avoids the zero-propagation problem and also, increases the convergence speed.

5. WEIGHTING FUNCTION IN DESIGNING QUASI-EQUIRIPPLE VARIABLE FRACTIONAL DELAY FILTERS

In defining the cost function for the design of equiripple variable fractal delay filter, the error is evaluated at some discrete set of frequency and delay. It is convenient to define a $p \times q$ error matrix **E** for p elements in the set of the delay range and q elements in the set of the frequency range. Each absolute error of these points is stored in this matrix. Fig. 1 depicts the absolute error of a non-equiripple variable fractional delay FIR filter which is designed with the WLS method. The updating matrix will start from the matrix **E**. By using the envelope of the matrix **E** is a good choice for the updating matrix. The updating function is now given by

$$W_{k+1}(w,d) = W_k(w,d)\, E(w,d) \tag{17}$$

By observing Fig.1, there are many ripples along the frequency axis, and almost no ripple along the delay axis. Because there are many ripples along the frequency axis, it's good to start by connecting the extremes in the direction of the frequency axis to form the envelope. This is, for the q elements in each row, to join the extremes in the elements to form the envelope. Let $E(w_0, d)$, $E(w_1, d)$,…, $E(w_q, d)$ denotes the q elements of one row (d now is fixed). The first thing is to find out all the extreme points from the q points. The definition of the extreme point is that, for some element m, if V(m) is the extreme, then $V(m)>V(m+1)$ and $V(m)>V(m-1)$. Note that the number of extreme point is less than q. So, for one extreme point at w_m and the next extreme point at w_n, to form the envelope, the values between these two extreme points just are the linear interpolation between the two extreme points. That is

$$E(w,d)\,|_{d\text{ is fixed}} = \frac{w-w_m}{w_n-w_m}E(w_n,d) + \frac{w_n-w}{w_n-w_m}E(w_m,d)$$
$$\text{for } 0 \le w_m < w < w_n \le \pi \quad \text{and} \quad 0 \le m < n \le q \tag{18}$$

Similarly, the updating function along the direction of the d (delay) axis is

$$E(w,d)\,|_{w\text{ is fixed}} = \frac{d-d_i}{d_j-d_i}E(w,d_j) + \frac{d_j-d}{d_j-d_i}E(w,d_i)$$
$$\text{for } 0 \le d_i < d < d_j \le 1 \quad \text{and} \quad 0 \le i < j \le p \tag{19}$$

This is shown in Fig. 2. One thing must be mentioned is that the first and the last element are also defined as extremes. Therefore, for the iteration was proceeded in column-wise, all the points locate at the first row and last row should be defined as extremes; and when the iteration was proceeded in row-wise, then the points at the first column and last column are all defined as extremes. After constructing the envelope for each row of the matrix **E**, process each column with the similar way.

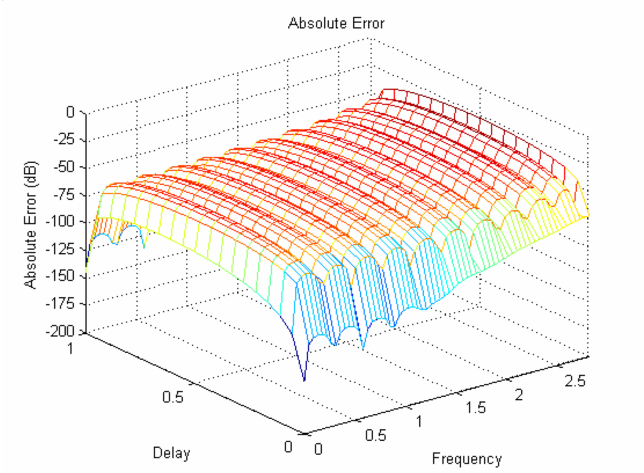

Fig. 1. The absolute error of the non-equiripple frequency response. The filter is designed with N=20, K=5, w_r=[0, 0.9π], d_r=[0, 1].

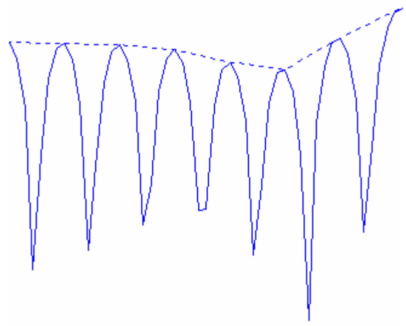

Fig. 2. The dotted line is the envelope of the solid line.

6. AN EXAMPLE

The method presented above is used to design a variable fractional delay filter with N=20 (the filter length is N+1), K=5, the frequency range w_r= [0, 0.9π] and the delay range d_r=[0, 1]. For the initial value of the weighting matrix, all elements are set to be unity.

The absolute error of the response designed with the WLS method for a non-equiripple fractional delay filter is shown in Fig1. It shows that the error is larger in higher frequency and the peak error is -28.6 dB. In Fig. 3, we can see that the peak absolute error decreases with the increase of iteration number. In general, the termination criterion can be, by defining a small positive ε, for the current kth iteration and the previous (k-1)th iteration,

$$\|E_k\| - \|E_{k-1}\| < \varepsilon \tag{20}$$

Or, the algorithm can be terminated after a pre-specified number of iterations. In our design, The required iteration number can be less than 9. The final result shown in Fig. 4 is taken at the tenth iteration and the peak error is read as -35.3 dB.

7. CONCLUSIONS

In this paper, a simple way to design a variable fractional delay filter is presented. It's straightforward and can be implemented in computer codes easily. Compare Fig. 1 and Fig. 4, it is easy to see the difference between the non-equiripple solution and the quasi-equiripple solution. The quasi-equiripple design is superior to the plain WLS (weighting function is unity and non-adapted) design in peak absolute error by 6.7 dB.

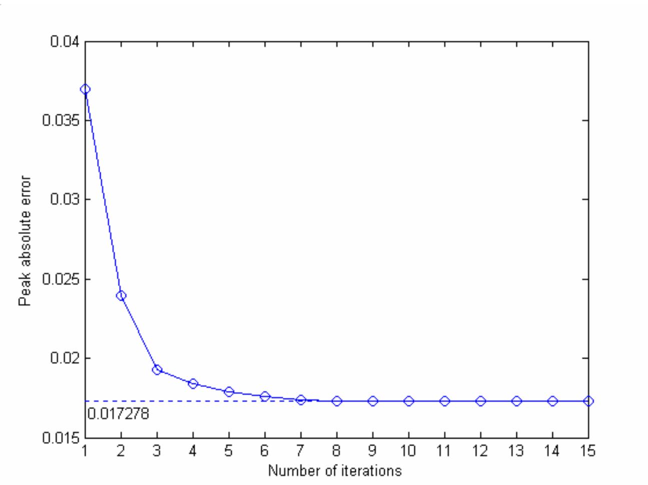

Fig. 3. Peak absolute error versus the number of iterations.

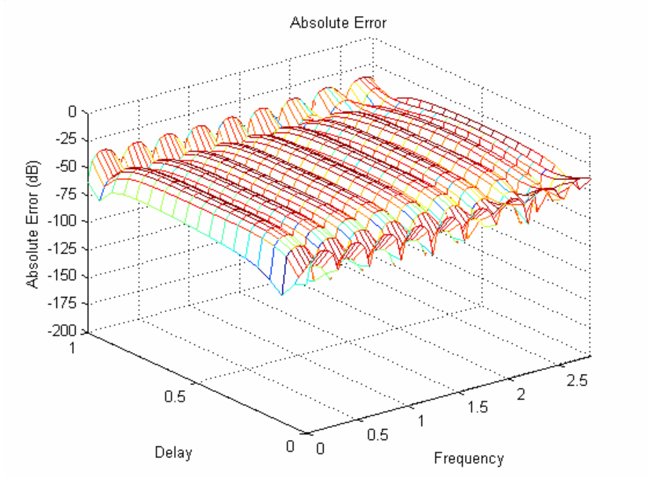

Fig. 4. The absolute error of the WLS equiripple design at the tenth iteration.

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