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CLOSED-FORM EXPRESSIONS OF THE TRUE CRAMER-RAO BOUND FOR PARAMETER ESTIMATION OF BPSK, MSK OR QPSK WAVEFORMS

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ABSTRACT

This paper addresses the stochastic Cramer-Rao bound (CRB) pertaining to the joint estimation of the carrier frequency offset, the carrier phase and the noise and data powers of binary phase-shift keying (BPSK), minimum shift keying (MSK) and quaternary phase-shift keying (QPSK) modulated signals corrupted by additive white circular Gaussian noise. Because the associated models are governed by simple Gaussian mixture distributions, an explicit expression of the Fisher Information matrix is given and an explicit expression for the stochastic CRB of these four parameters are deduced. Refined expressions for low and high-SNR are presented as well. Finally, our proposed analytical expressions are numerically compared with the approximate expressions previously given in the literature.

1. INTRODUCTION

The stochastic Cramer-Rao bound (CRB) is a well known lower bound on the variance of any unbiased estimate, and as such, serves as useful benchmark for practical estimators. Unfortunately, the evaluation of this CRB is mathematically quite difficult when the observed signal contains, in addition to the parameters to be estimated, random discrete data and random noise. A typical example of such a situation that has been studied by many authors (see e.g., [1] and the references therein) is the observation of noisy linearly modulated waveforms that are a function of deterministic parameters such that the time delay, the carrier frequency offset, the carrier phase, noise and data powers, as well as the data symbol sequence. Because the analytical computation of this CRB has been considered to be unfeasible, a modified CRB (MCRB) which is much simpler to evaluate than the true CRB has been introduced in [2]. But this MCRB may not be as tight as the true CRB [3] for joint estimation of all parameters. To circumvent this difficulty, asymptotic expressions at low [4] or high [5] signal-to-noise ratio (SNR) have been investigated. But unfortunately, these asymptotic expressions do not apply at moderate SNR, for which only combined analytical/numerical (see e.g., [5, 6, 1]) approaches are available until now.

In this paper, we investigate an analytical expression of the stochastic CRB associated with the joint estimation of the carrier frequency offset, the carrier phase and the noise and data powers of BPSK, QPSK or MSK modulated signals corrupted by additive white circular Gaussian noise, which is valid for arbitrary SNR. This paper is organized as follows. After formulating the problem in Section 2, an explicit expression of the Fisher information matrix (FIM) associated with all the deterministic parameters is given in Section 3. Because the carrier frequency offset and the carrier phase parameters are decoupled from the signal noise and data powers parameters, simple explicit expressions for the stochastic CRB of these four parameters are deduced. Refined expressions for low and high-SNR are presented as well. Finally, in Section 4, our proposed analytical expressions are numerically compared with the previously given approximate expressions.

2. PROBLEM FORMULATION

Consider BPSK, QPSK or MSK modulated signals. The received signals are bandpass filtered and after down-shifting the signal to baseband, the in-phase and quadrature components are paired to obtain complex signals. We assume Nyquist shaping and ideal sample timing so that the intersymbol interference at each symbol spaced sampling instance can be ignored. In the presence of frequency offset and carrier phase, the signals at the output of the matched filters yield the observation vector $\mathbf{y} = (y_{k_0}, ..., y_{k_0+K-1})$, with

$$y_k = a s_k e^{i2\pi k\nu} e^{i\phi} + n_k,$$

for $k = k_0, ..., k_0 + K - 1$. $\{s_k\}$ is a sequence of independent identically distributed (IID) data symbols taking values ± 1 [resp. $\pm\sqrt{2}/2 \pm i\sqrt{2}/2$] with equal probabilities for BPSK [resp., QPSK] modulations and for MSK modulations are defined by $s_{k+1} = i s_k c_k$ where c_k is a sequence of independent BPSK symbols with equal probabilities where the original value s_{k_0} remains unspecified in the set $\{+1, +i, -1, -i\}$. The deterministic unknown parameters a, ν and ϕ represent the amplitude, the carrier frequency offset normalized to the symbol rate and the carrier phase at $k = 0$. Finally, the sequence $\{n_k\}$ consists of IID zero-mean complex circular Gaussian noise variables[1] of variance σ^2. The symbols s_k are assumed to be independent from n_k.

If no a priori information is available concerning the transmitted symbols, the distribution of $\mathbf{y}$ is parameterized by $\theta \stackrel{\text{def}}{=} (\nu, \phi, a, \sigma)$. We note that the MSK modulation is modelled equivalently (see e.g., [7]) by $s_k = i^{k-k_0} b_k s_{k_0}$ where b_k is another sequence of independent BPSK symbols $\{-1, +1\}$ with equal probabilities. Consequently, similarly to the BPSK and QPSK modulations, $(y_k)_{k=k_0,...,k_0+K-1}$ are

[1] Note that many papers consider the parameters a^2 and σ^2 denoted usually as the symbol energy E_s and the noise power spectral density N_0 as known. They usually suppose a unit variance for the noise and use the ratio $\varepsilon \stackrel{\text{def}}{=} (E_s/N_0)^{1/2}$ as the modulation amplitude, but in practice these two parameters are unknown.

independently non-identically distributed along the following mixed circular Gaussian distribution:

$$p(y_k;\theta) = \frac{1}{L\pi\sigma^2}\sum_{l=1}^{L}\exp\left(-\frac{|y_k - a s_{l,k} e^{i2\pi k\nu} e^{i\phi}|^2}{\sigma^2}\right), \quad (1)$$

with $s_{l,k} = \pm 1$ $(L=2)$, $s_{l,k} = \pm\sqrt{2}/2 \pm i\sqrt{2}/2$ $(L=4)$ or $s_{l,k} = i^{k-k_0} b_l s_{k_0}$ with $b_l = \pm 1$ $(L=2)$ associated with BPSK, QPSK or MSK modulations, respectively.

3. STOCHASTIC CRB: ANALYTICAL RESULTS

3.1 General closed-form expression

Using the independence of the random variables y_k, the Fisher information matrices (FIM) is given (elementwise) by:

$$(\mathbf{I}_F)_{i,j} = -\sum_{k=k_0}^{k_0+K-1} \mathrm{E}\left(\frac{\partial^2 \ln p(y_k;\theta)}{\partial\theta_i \partial\theta_j}\right) \qquad i,j = 1,\ldots,4, \quad (2)$$

where the PDF's (1) take the following forms:

$$p(y_k;\theta) = \frac{1}{\pi\sigma^2}\exp\left(-\frac{|y_k|^2 + a^2}{\sigma^2}\right) c(y_k),^2$$

where

$$\begin{aligned} c(y_k) &= \cosh\left(\frac{a}{\sigma^2}\, g_1(y_k)\right) \\ &= \cosh\left(\frac{a}{\sigma^2\sqrt{2}}\, g_1(y_k)\right)\cosh\left(\frac{a}{\sigma^2\sqrt{2}}\, g_2(y_k)\right) \\ &= \cosh\left(\frac{a}{\sigma^2}\, g_3(y_k)\right) \end{aligned}$$

for the BPSK, QPSK and MSK modulations respectively, with $g_1(y_k) \stackrel{\text{def}}{=} 2\Re(e^{i2\pi k\nu} e^{i\phi} y_k^*)$, $g_2(y_k) \stackrel{\text{def}}{=} 2\Im(e^{i2\pi k\nu} e^{i\phi} y_k^*)$ and $g_3(y_k) \stackrel{\text{def}}{=} 2\Re(i^{k-k_0} e^{i2\pi k\nu} e^{i\phi} s_{k_0} y_k^*)$. Extending the approach used in [8] for the parameters a and σ only and in [9] for the direction of arrival (DOA) parameters, the following lemma is proved in Appendix A:

Lemma 1 *The parameter $\theta = (\nu,\phi,a,\sigma)$ is partitioned in two decoupled parameters (ν,ϕ) and (a,σ) in the FIM associated with the BPSK, QPSK and MSK modulations:*

$$\begin{aligned} \mathbf{I}_{\mathrm{BPSK}} &= \mathbf{I}_{\mathrm{MSK}} = \begin{bmatrix} \mathbf{I}_{\mathrm{B}}^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{\mathrm{B}}^{(2)} \end{bmatrix} \\ \mathbf{I}_{\mathrm{QPSK}} &= \begin{bmatrix} \mathbf{I}_{\mathrm{Q}}^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{\mathrm{Q}}^{(2)} \end{bmatrix} \end{aligned}$$

with

$$\begin{aligned} \mathbf{I}_{\mathrm{B}}^{(1)} &= 2\rho^2(1-f_1(\rho)) \\ &\quad \begin{bmatrix} (2\pi)^2\sum_{k=k_0}^{k_0+K-1} k^2 & 2\pi\sum_{k=k_0}^{k_0+K-1} k \\ 2\pi\sum_{k=k_0}^{k_0+K-1} k & K \end{bmatrix} \\ \mathbf{I}_{\mathrm{B}}^{(2)} &= 2K\frac{\rho}{a^2}\begin{bmatrix} 1-f_2(\rho) & 2\sqrt{\rho} f_2(\rho) \\ 2\sqrt{\rho} f_2(\rho) & 2(1-2\rho f_2(\rho)) \end{bmatrix} \\ \mathbf{I}_{\mathrm{Q}}^{(1)} &= 2\rho^2(1-(1+\rho) f_1(\tfrac{\rho}{2})) \\ &\quad \begin{bmatrix} (2\pi)^2\sum_{k=k_0}^{k_0+K-1} k^2 & 2\pi\sum_{k=k_0}^{k_0+K-1} k \\ 2\pi\sum_{k=k_0}^{k_0+K-1} k & K \end{bmatrix} \\ \mathbf{I}_{\mathrm{Q}}^{(2)} &= 2K\frac{\rho}{a^2}\begin{bmatrix} 1-f_2(\frac{\rho}{2}) & 2\sqrt{\rho} f_2(\frac{\rho}{2}) \\ 2\sqrt{\rho} f_2(\frac{\rho}{2}) & 2(1-2\rho f_2(\frac{\rho}{2})) \end{bmatrix} \end{aligned}$$

where ρ is the SNR $\frac{a^2}{\sigma^2}$ and f_1 and f_2 are the following decreasing functions of ρ:

$$f_1(\rho) \stackrel{\text{def}}{=} \frac{2e^{-\rho}}{\sqrt{2\pi}}\int_0^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\cosh(u\sqrt{2\rho})}du,$$

$$f_2(\rho) \stackrel{\text{def}}{=} \frac{2e^{-\rho}}{\sqrt{2\pi}}\int_0^{+\infty} \frac{u^2 e^{-\frac{u^2}{2}}}{\cosh(u\sqrt{2\rho})}du.$$

The determinants of $\mathbf{I}_{\mathrm{B}}^{(1)}$ and $\mathbf{I}_{\mathrm{Q}}^{(1)}$ do not depend on the time k_0 at which the first sample is taken and consequently the CRB for the frequency does not depend on it either, but the CRB for the phase does. The minimum value for this CRB is attained for $k_0 = -(K-1)/2$. This particular choice of k_0 renders $\mathbf{I}_{\mathrm{B}}^{(1)}$ and $\mathbf{I}_{\mathrm{Q}}^{(1)}$ diagonal and we obtain in this case the following result, where the MCRB are straightforwardly derived from [2]:

$$\mathrm{MCRB}(\theta_i) = \frac{1}{\mathrm{E}\left(\frac{\partial^2 \ln p(\mathbf{y}/\mathbf{s};\theta)}{\partial\theta_i^2}\right)}, \qquad i = 1,\ldots,4.$$

Result 1 *The CRB for the joint estimation of the parameters (ν,ϕ,a,σ) associated with the BPSK and MSK modulations are given by:*

$$\begin{aligned} \mathrm{CRB}(\nu) &= \frac{6}{(2\pi)^2 K(K^2-1)\rho(1-f_1(\rho))} \\ &= \mathrm{MCRB}(\nu)\left(\frac{1}{1-f_1(\rho)}\right) \quad (3) \\ \mathrm{CRB}(\phi) &= \frac{1}{2K\rho(1-f_1(\rho))} \\ &= \mathrm{MCRB}(\phi)\left(\frac{1}{1-f_1(\rho)}\right) \quad (4) \\ \mathrm{CRB}(a) &= \frac{a^2(1-2\rho f_2(\rho))}{2K\rho(1-f_2(\rho)-2\rho f_2(\rho))} \\ &= \mathrm{MCRB}(a)\left(\frac{1-2\rho f_2(\rho)}{1-f_2(\rho)-2\rho f_2(\rho)}\right) \quad (5) \\ \mathrm{CRB}(\sigma) &= \frac{a^2(1-f_2(\rho))}{4K\rho(1-f_2(\rho)-2\rho f_2(\rho))} \\ &= \mathrm{MCRB}(\sigma)\left(\frac{1-2\rho f_2(\rho)}{1-f_2(\rho)-2\rho f_2(\rho)}\right). \quad (6) \end{aligned}$$

[2] Note that this product form does not extend to arbitray QAM modulation (see e.g., [6, rel. (41)] for the 16QAM modulation.

The CRBs associated with the QPSK modulation are obtained by replacing $f_1(\rho)$ and $f_2(\rho)$ by, respectively, $(1+\rho)f_1(\frac{\rho}{2})$ and $f_2(\frac{\rho}{2})$ in (3), (4), (5) *and* (6).

Consequently, for high SNR, the asymptotic CRB coincides with the MCRB. This extends a property proved in [3] for a scalar parameter only.

3.2 Low-SNR expression

For low SNR, $f_1(\rho)$ and $f_2(\rho)$ approach 1. We resort to a Taylor series expansion of these functions obtained by expanding $e^{-\rho}$ and $\cosh(u\sqrt{2\rho})$ around $\rho=0$. Then, using the values $(2n)!/2^n n!$ of the moments of order $2n$ of zero-mean unit variance Gaussian random variables, we obtain after tedious, but straightforward algebraic manipulations:

$$\begin{aligned} f_1(\rho) &= 1-2\rho+4\rho^2-\frac{40}{3}\rho^3+\frac{208}{3}\rho^4+o(\rho^4), \\ f_2(\rho) &= 1-4\rho+16\rho^2-\frac{256}{3}\rho^3+\frac{12544}{21}\rho^4+o(\rho^4). \end{aligned}$$

Inserting these expansions in Result 1 allows us to prove the following result:

Result 2 *The CRB for the joint estimation of the parameters (ν,ϕ,a,σ) associated with the BPSK, MSK and QPSK modulations are given for low SNR by:*

$$\begin{aligned} \mathrm{CRB}(\nu) &= \frac{6}{(2\pi)^2K(K^2-1)}\frac{L!}{L^2\rho^L}(1+L\rho+o(\rho)) \\ &= \mathrm{MCRB}(\nu)\frac{L!}{L^2\rho^{L-1}}(1+L\rho+o(\rho)) \qquad (7) \\ \mathrm{CRB}(\phi) &= \frac{1}{2K}\frac{L!}{L^2\rho^L}(1+L\rho+o(\rho)) \\ &= \mathrm{MCRB}(\phi)\frac{L!}{L^2\rho^{L-1}}(1+L\rho+o(\rho)) \qquad (8) \\ \mathrm{CRB}(a) &= \frac{a^2}{K\alpha_L\rho^L}(1+L\rho+o(\rho)) \\ &= \mathrm{MCRB}(a)\frac{2}{\alpha_L\rho^{L-1}}(1+L\rho+o(\rho)) \qquad (9) \\ \mathrm{CRB}(\sigma) &= \frac{a^2}{K\beta_L\rho^{L-1}}(1+\gamma_L\rho^{3-L/2}+o(\rho^{3-L/2})) \\ &= \mathrm{MCRB}(\sigma)\frac{4}{\beta_L\rho^{L-2}}(1+\gamma_L\rho^{3-L/2}+o(\rho^{3-L/2})), \qquad (10) \end{aligned}$$

with $L=2$ [resp. 4] for the BPSK and the MSK [resp. QPSK] modulation and $\alpha_2=4$, $\alpha_4=16/3$, $\beta_2=2$, $\beta_4=16/3$, $\gamma_2=-16/3$ and $\gamma_4=4$.

We note that (7) and (8) for BPSK and QPSK modulations are refinements of the expressions of $\mathrm{CRB}(\nu)$ and $\mathrm{CRB}(\phi)$ given in [4].

3.3 High-SNR expression

For high SNR, the MCRB approaches the CRB at the same speed that $f_1(\rho)$ and $f_2(\rho)$ approach 0. Because we prove in Appendix B that these functions are bounded above by $\frac{e^{-\rho}}{\sqrt{\pi\rho}}$ and more precisely that $f_1(\rho)/\frac{e^{-\rho}\ln 2}{\sqrt{\pi p}}$ tends to 1 when ρ tends to ∞, the CRB are practically equal to the MCRB for moderate SNR. For example: $\rho=2$ (3dB) [resp. $\rho=4$ (6dB)] gives the upper bound 0.05 [resp. 0.005] for $f_1(\rho)$ and $f_2(\rho)$ and consequently the ratios CRB/MCRB are of the same order of magnitude from these values of SNR.

4. NUMERICAL RESULTS

The analytical Result 1 is numerically compared with the approximations given in Result 2 and to the approximations given in [4] for $\mathrm{CRB}(\nu)$ and $\mathrm{CRB}(\phi)$ of BPSK and QPSK modulations at low SNR.

In these conditions, we see good agreement between the numerical values derived from Results 1 and 2 in a large domain of low SNR. Furthermore, we note that the ratio CRB/MCRB is unbounded except for the noise power of BPSK and MSK modulations for which it tends to 2 when the SNR tends to 0.

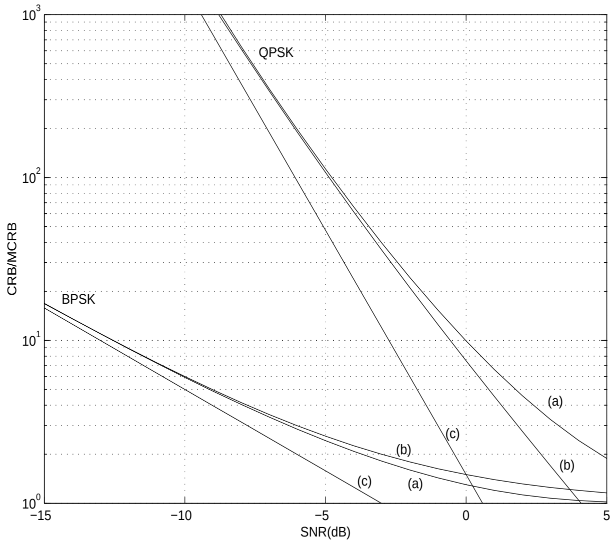

Fig.1 Ratio $\mathrm{CRB}(\nu)/\mathrm{MCRB}(\nu)=\mathrm{CRB}(\phi)/\mathrm{MCRB}(\phi)$ at low SNR: (a) exact value given by (3), (4), (b) approximate value given by (7), (8), (c) approximate value given in [4].

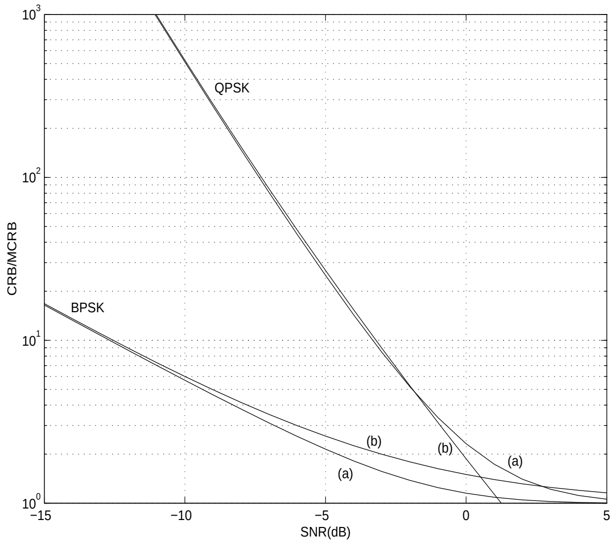

Fig.2 Ratio $\mathrm{CRB}(a)/\mathrm{MCRB}(a)$ at low SNR: (a) exact value given by (5), (b) approximate value given by (9).

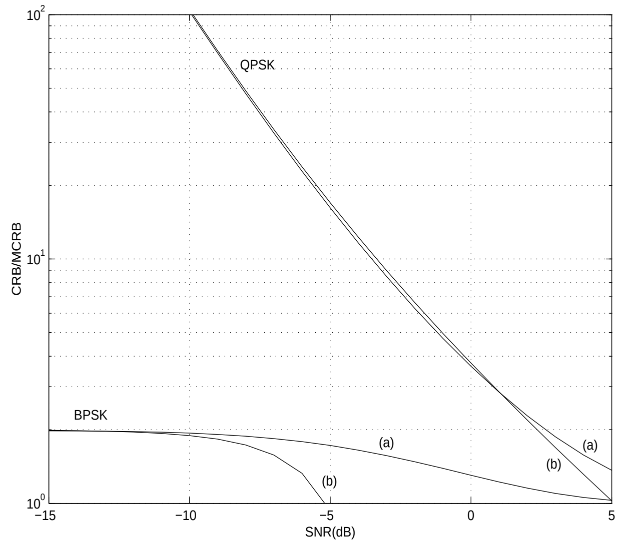

Fig.3 Ratio CRB(σ)/MCRB(σ) at low SNR: (a) exact value given by (6), (b) approximate value given by (10).

A. APPENDIX: PROOF OF LEMMA 1

To evaluate the FIM (2) for the BPSK modulation, we take partial derivatives as follows:

$$
\begin{aligned}
\frac{\partial^2 \ln p(y_k;\theta)}{\partial \phi^2} &= \frac{a^2 g_2^2(y_k)}{\sigma^4}\frac{1}{\cosh^2\left(\frac{ag_1(y_k)}{\sigma^2}\right)} \\
&- \frac{ag_1(y_k)}{\sigma^2}\tanh\left(\frac{ag_1(y_k)}{\sigma^2}\right) \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial \nu^2} &= (2\pi k)^2 \frac{\partial^2 \ln p(y_k;\theta)}{\partial \phi^2} \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial \phi \partial \nu} &= (2\pi k) \frac{\partial^2 \ln p(y_k;\theta)}{\partial \phi^2} \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial a^2} &= -\frac{2}{\sigma^2} + \frac{g_1^2(y_k)}{\sigma^4}\frac{1}{\cosh^2\left(\frac{ag_1(y_k)}{\sigma^2}\right)} \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial \sigma^2} &= \frac{2\sigma^2 - 6(a^2+|y_k|^2)}{\sigma^4} \\
&+ \frac{4a^2 g_1^2(y_k)}{\sigma^6}\frac{1}{\cosh^2\left(\frac{ag_1(y_k)}{\sigma^2}\right)} \\
&+ \frac{6ag_1(y_k)}{\sigma^4}\tanh\left(\frac{ag_1(y_k)}{\sigma^2}\right) \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial a \partial \sigma} &= \frac{4a}{\sigma^3} - \frac{2ag_1^2(y_k)}{\sigma^5}\frac{1}{\cosh^2\left(\frac{ag_1(y_k)}{\sigma^2}\right)} \\
&- \frac{2g_1(y_k)}{\sigma^3}\tanh\left(\frac{ag_1(y_k)}{\sigma^2}\right) \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial a \partial \phi} &= -\frac{ag_1(y_k)g_2(y_k)}{\sigma^4}\frac{1}{\cosh^2\left(\frac{ag_1(y_k)}{\sigma^2}\right)} \\
&- \frac{g_2(y_k)}{\sigma^2}\tanh\left(\frac{ag_1(y_k)}{\sigma^2}\right) \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial a \partial \nu} &= (2\pi k) \frac{\partial^2 \ln p(y_k;\theta)}{\partial a \partial \phi} \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial \sigma \partial \phi} &= \frac{2a^2 g_1(y_k)g_2(y_k)}{\sigma^5}\frac{1}{\cosh^2\left(\frac{ag_1(y_k)}{\sigma^2}\right)} \\
&+ \frac{2ag_2(y_k)}{\sigma^3}\tanh\left(\frac{ag_1(y_k)}{\sigma^2}\right) \\
\frac{\partial^2 \ln p(y_k;\theta)}{\partial \sigma \partial \nu} &= (2\pi k) \frac{\partial^2 \ln p(y_k;\theta)}{\partial \sigma \partial \phi}.
\end{aligned}
$$

Using the regularity condition $\frac{\partial}{\partial \theta_i}\int p(y_k;\theta)dy_k = \int \frac{\partial p(y_k;\theta)}{\partial \theta_i}dy_k$ which is fulfilled for finite mixtures of Gaussian distributions, the following property holds: $\mathrm{E}\left(\frac{\partial \ln p(y_k;\theta)}{\partial a}\right) = 0$. With $\frac{\partial \ln p(y_k;\theta)}{\partial a} = -\frac{2a}{\sigma^2} + \frac{g_1(y_k)}{\sigma^2}\tanh\left(\frac{ag_1(y_k)}{\sigma^2}\right)$, we obtain

$$\mathrm{E}\left(g_1(y_k)\tanh\left(\frac{ag_1(y_k)}{\sigma^2}\right)\right) = 2a.$$

This identity enables us to straightforwardly derive the terms of $\mathbf{I}_{\mathrm{B}}^{(2)}$ thanks to the definition of the function $f_2(\rho) = \mathrm{E}\left(\frac{g_1^2(y_k)}{2\sigma^2}\frac{1}{\cosh^2\left(\frac{ag_1(y_k)}{\sigma^2}\right)}\right)$, where the r.v. $g_1(y_k)$ is equally weighted mixed Gaussian ($\mathcal{N}(-2a;2\sigma^2)$ and $\mathcal{N}(+2a;2\sigma^2)$).

To evaluate $\mathbf{I}_{\mathrm{B}}^{(1)}$, we note that $g_1(y_k) = 2as_k + \left(e^{i2\pi k\nu}e^{i\phi}n_k^* + e^{-i2\pi k\nu}e^{-i\phi}n_k\right)$ and $g_2(y_k) = -i\left(e^{i2\pi k\nu}e^{i\phi}n_k^* - e^{-i2\pi k\nu}e^{-i\phi}n_k\right)$. Because s_k and n_k are independent and the two Gaussian r.v. $e^{i2\pi k\nu}e^{i\phi}n_k^* + e^{-i2\pi k\nu}e^{-i\phi}n_k$ and $e^{i2\pi k\nu}e^{i\phi}n_k^* - e^{-i2\pi k\nu}e^{-i\phi}n_k$ are uncorrelated and therefore independent, the three r.v. s_k, $e^{i2\pi k\nu}e^{i\phi}n_k^* + e^{-i2\pi k\nu}e^{-i\phi}n_k$ and $e^{i2\pi k\nu}e^{i\phi}n_k^* - e^{-i2\pi k\nu}e^{-i\phi}n_k$ are collectively independent and thus $g_1(y_k)$ and $g_2(y_k)$ are independent. This implies that the parameters (ν,ϕ) and (a,σ) are decoupled in the FIM. Using the definition of the function $f_1(\rho) = \mathrm{E}\left(\frac{1}{\cosh^2\left(\frac{ag_1(y_k)}{\sigma^2}\right)}\right)$, the terms of $\mathbf{I}_{\mathrm{B}}^{(1)}$ are derived.

For the MSK modulation, the derivations follow the same lines, replacing $g_1(y_k)$ by $g_3(y_k)$.

Finally for the QPSK modulation, evaluating the partial derivatives $\frac{\partial^2 \ln p(y_k;\Theta)}{\partial \theta_i \partial \theta_j}$ and taking their expectation are derived in the same way, provided the log-likelihoods associated with $g_1(y_k)$ and $g_2(y_k)$ are gathered, and the hypothesis of independence of $\Re(s_k)$ and $\Im(s_k)$ is taken into account.

B. APPENDIX

For high SNR, using the inequality

$$\frac{1}{\cosh(\mathrm{u}\sqrt{2\rho})} < 2e^{-u\sqrt{2\rho}},$$

we obtain after simple algebraic manipulations:

$$f_1(\rho) < 4Q(\sqrt{2\rho}) \qquad (11)$$

and

$$f_2(\rho) < 4\left((2\rho+1))Q(\sqrt{2\rho}) - \frac{\sqrt{2\rho}}{\sqrt{2\pi}}e^{-\rho}\right), \qquad (12)$$

where Q(x) is the error function $\int_x^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du$ classically bounded above by $\frac{1}{x\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. Applying this upper bound in (11) and (12) gives: $f_1(\rho) < \frac{e^{-\rho}}{\sqrt{\pi\rho}}$ and $f_2(\rho) < \frac{e^{-\rho}}{\sqrt{\pi\rho}}$. To specify the upper bound of $f_1(\rho)$, we use the following expansion

$$\begin{aligned}\frac{1}{\cosh(\mathrm{u}\sqrt{2\rho})} &= 2e^{-u\sqrt{2\rho}}(1+e^{-u\sqrt{2\rho}})^{-1} \\ &= 2\sum_{k=0}^{+\infty}(-1)^k e^{-(k+1)u\sqrt{2\rho}}.\end{aligned}$$

Inserting this into $f_1(\rho)$, we obtain after simple algebraic manipulations the following alternating expansion:

$$f_1(\rho) = 4\sum_{k=0}^{+\infty}(-1)^k e^{\rho[(k+1)^2-1]}\mathrm{Q}[(k+1)\sqrt{2\rho}]f_2(\rho). \quad (13)$$

Using the standard bounds $\frac{1}{x\sqrt{2\pi}}(1-\frac{1}{x^2})e^{-\frac{x^2}{2}} \leq \mathrm{Q}(x) \leq \frac{1}{x\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $\ln 2 = -\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ in (13) proves after simple algebraic manipulations that $f_1(\rho)/\frac{e^{-\rho}\ln 2}{\sqrt{\pi p}}$ tends to 1 when ρ tends to ∞.

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