Blind Identifiability of Class of Nonlinear Instantaneous ICA Models

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ABSTRACT

The identifiability/separability of nonlinear instantaneous ICA models is considered. The identifiability proof is constructed for the class of nonlinearities satisfying addition theorem. Addition theorem covers wide variety of nonlinear mixing systems of engineering interest. An algorithm for separating such nonlinear mixtures is presented and the feasibility of the approach is demonstrated.

1 Introduction

In this paper, we address the problem of blind identifiability of nonlinear instantaneous ICA models. No general identifiability conditions exist for such models. Therefore, one needs to construct identifiability proofs for different classes of nonlinearities and sources. Obviously it is valuable to show identifiability to such classes of nonlinear functions and source distributions that cover wide variety of situations of engineering interest. In this paper we concentrate on a class of nonlinearities satisfying so-called addition theorem that covers many nonlinearities of practical interest. The addition theorem has been previously [KLR73b] applied to extend the Darmois-Skitovich theorem, which can be viewed to lie in the heart of the identifiability of the linear ICA model. Here we use the addition theorem directly to the nonlinear ICA problem. The application is two folded. First, it is shown that some natural restrictions on structure of nonlinear mixing necessarily lead to models for which the addition theorem holds. Second, the identifiability of these nonlinear models is proved. Also a generic algorithm for blind separation of such nonlinear ICA models is presented. Examples demonstrating the feasibility of the approach are provided.

This paper is organized as follows. In Section 2, the instantaneous ICA model and the blind separation problem are described in a general form. In Section 3, a class of nonlinear ICA models is considered and an identifiability proof is constructed. An algorithm for separating such ICA models is presented. Finally, Section 4 demonstrates the feasibility of the approach.

2 System model

A general instantaneous Independent Component Analysis (ICA) model with instantaneous mixing may be described by the equation

$$\mathbf{X} = \mathcal{F}(\mathbf{S}), \tag{1}$$

where the sources $[S_1, S_2, \ldots, S_m]^T = \mathbf{S}$ are real valued independent non-degenerate random variables, $\mathcal{F}$ is a measurable function $\mathbb{R}^m \to \mathbb{R}^p$, $p \geq 2$, and $\mathbf{X} = [X_1, X_2, \ldots, X_p]^T$ is the observed random vector. The separation problem consists of finding a transform $\mathcal{G}$ such that each component of $\mathbf{Y} = \mathcal{G}(\mathcal{F}(\mathbf{S}))$ can be written as

$$Y_i = \kappa_i(S_{\sigma(i)}), \quad i = 1, \ldots, m, \tag{2}$$

where σ is a permutation on $\{1, 2, \ldots, m\}$ and the function κ_i represents residual distortion. The function $\mathcal{G}$ should be found using only the observed mixture $\mathbf{X}$ and the assumption that the source variables are statistically independent. In general,it can be shown that independence cannot insure separation [HP99, TJ99], i.e. there exists infinite number of functions $\mathcal{G}$ such that components of $\mathcal{G}(\mathbf{X})$ are independent and are not of the form (2). Thus it is necessary to restrict the model to contain only variables $\mathbf{S}$ and functions $\mathcal{F}$ from certain classes of variables and functions. Then we may say that the separation problem has a solution, or that the model (1) is *separable* up to distortion κ_i, if the independence of the components of $\mathbf{Y} = \mathcal{G}(\mathbf{X})$ implies that $\mathbf{Y}$ is of the form (2).

3 Identifiability of a class of Nonlinear ICA models

The traditional *linear ICA model* is obtained from (1) by restricting $\mathcal{F}$ to be linear, i.e. $\mathcal{F}(\mathbf{S}) = A\mathbf{S} + \mathbf{c}$ for some $p \times m$ matrix A and $p \times 1$ vector $\mathbf{c}$ of constants. Since independence of two random variables is preserved by adding constants to both variables (implying necessary location distortion to the model), it is easy to see ([KLR73a], Lemma 10.2.3) that without loss of generality we can omit the constant and write the linear model as

$$\mathbf{X} = A\mathbf{S}. \tag{3}$$

This model has been extensively studied over the last few years (see e.g. [HKO01]). The original proof of separability for the case $p = m$ was given in [Com94] and some extensions were presented in [CL96]. The model is found to be identifiable up to scaling and location distortion if $p \geq m$ and at most one of the source variables has a Gaussian distribution. The cited proofs make an additional assumption that the sources have finite variances, a condition that can apparently be removed [KLR73a].

As noted in Section 1, a general nonlinear ICA model is not separable. It is therefore necessary to somehow restrict the function $\mathcal{F}$ and/or the source distributions to certain classes in order to find the solution. We now consider some natural restrictions on the type and structure of the nonlinear operation. These conditions make perhaps surprisingly the model separable as will be shown in the end of this section.

Analogously to the linear model, a justified condition for the function is that the components of $\mathcal{F}$ are scalable in the sense that they can be constructed from lower dimensional functions. Thus it is natural to consider measurable functions $\mathcal{F}_2(u, v)$ for the variables on some (possibly infinite) interval (α, β) of the real axis. Suppose further that the operation is closed, i.e. $\mathcal{F}_2(u, v) \in (\alpha, \beta)$ and that the following conditions are satisfied:

(i) $\mathcal{F}_2(u, v)$ is continuous separately both for u and v.

(ii) $\mathcal{F}_2$ is commutative, i.e. $\mathcal{F}_2(u, v) = \mathcal{F}_2(v, u)$ for all $u, v \in (\alpha, \beta)$.

(iii) $\mathcal{F}_2$ is associative, i.e. $\mathcal{F}_2(\mathcal{F}_2(u, v), w) = \mathcal{F}_2(u, \mathcal{F}_2(v, w))$ for all $u, v, w \in (\alpha, \beta)$.

(iv) There exist an identity element $e \in (\alpha, \beta)$, i.e. $\mathcal{F}_2(e, u) = u$ for each $u \in (\alpha, \beta)$.

(v) For each $u \in (\alpha, \beta)$ there exists an inverse element $u^{-1} \in (\alpha, \beta)$, i.e. $\mathcal{F}_2(u, u^{-1}) = e$.

The conditions (ii) and (iii) reflect the idea that in the model (1) the order of the mixing should be immaterial. The conditions (iv) and (v) describe the situation that a signal value does not affect the other signals and that a signal value cancels out the other signal value, respectively. It is also noted that the conditions (ii)–(v) make the function (operation) $\mathcal{F}_2$ an Abelian (commutative) group. The operation $\mathcal{F}_2(\cdot, \cdot)$ is from here on interchangeably denoted by $\circ$ for brevity.

Under the conditions stated above it is known from the theory of functional equations [Acz66] that there exists strictly monotonic continuous function f: $\mathbb{R} \to (\alpha, \beta)$ such that

$$f(x + y) = \mathcal{F}_2\big(f(x), f(y)\big). \tag{4}$$

Operator $u \circ v$	Function $f(x)$
$u + v$	cx
uv	e^{cx}
$u + v + uv$	$e^{cx} - 1$
$\frac{uv}{u+v}$	$\frac{c}{x}$
$\frac{u+v+2uv}{1-uv}$	$\frac{cx}{1-cx}$
$\frac{u+v-2uv}{1-uv}$	$\frac{-cx}{1-cx}$
$uv - \sqrt{1-u^2}\sqrt{1-v^2}$	$\cos(cx)$
$uv + \sqrt{u^2-1}\sqrt{v^2-1}$	$\cosh(cx)$
$\frac{u+v}{1-uv}$	$\tan(cx)$
$\frac{uv-1}{u+v}$	$\cot(cx)$
$\frac{u+v}{1+uv/c^2}$	$c\tanh(bx)$
$\frac{u+v-2uv\cos(c)}{1-uv}$	$\frac{\sin(bx)}{\sin(bx+c)}$
$\frac{u+v-1}{2u+2v-2uv-1}$	$\frac{1}{1+\tan(cx)}$
$\frac{u+v-2uv}{1-2uv}$	$\frac{1}{1+\cot(cx)}$
$\frac{u+v-2uv\cosh(c)}{1-uv}$	$\frac{\sinh(bx)}{\sinh(bx+c)}$
$\frac{u+v+2uv\cosh(c)}{1-uv}$	$\frac{-\sinh(bx)}{\sinh(bx+c)}$

Table 1: Examples [Acz66] of functions satisfying the addition theorem over reals. All operations are defined on the intervals where the corresponding function is continuous and strictly monotonic. Letters b and c denote arbitrary constants.

Such functions are said to satisfy the *addition theorem* [Acz66]. Some examples are given in Table 1. Also the converse [Acz66] to the theorem is true. Namely if any continuous function satisfies (4) for some $\mathcal{F}_2$, then necessarily $\mathcal{F}_2$ is a continuous group on some open interval.

Since f is strictly monotonic, it has the inverse function f^{-1}. Using (4) we have

$$f(x + x + \cdots + x) = f(x) \circ f(x) \circ \cdots \circ f(x) \triangleq n \star f(x),$$

which gives a new operation $\star$ when n is an integer. This is extended to all reals c by defining

$$c \star f(x) \triangleq f(cx). \tag{5}$$

In [KLR73b] the $\circ$ and $\star$ operators were used to construct nonlinear forms to extend the Darmois-Skitovich theorem for this nonlinear case. Here we use these operators to define a class of nonlinear models. Assuming the source variables S_i take values in (α, β), a nonlinear ICA model in (1) can be written as

$$\mathbf{X} = \begin{bmatrix} \mathcal{F}^{(1)}(S_1, S_2, \ldots, S_m) \\ \mathcal{F}^{(2)}(S_1, S_2, \ldots, S_m) \\ \vdots \\ \mathcal{F}^{(p)}(S_1, S_2, \ldots, S_m) \end{bmatrix} = \begin{bmatrix} a_{11} \star S_1 \circ a_{12} \star S_2 \circ \cdots \circ a_{1m} \star S_m \\ a_{21} \star S_1 \circ a_{22} \star S_2 \circ \cdots \circ a_{2m} \star S_m \\ \vdots \\ a_{p1} \star S_1 \circ a_{p2} \star S_2 \circ \cdots \circ a_{pm} \star S_m \end{bmatrix}. \tag{6}$$

Theorem 1. *Suppose the model (6) holds such that random variables* $f^{-1}(S_i)$, $i = 1, 2, \ldots, m$ *are non-Gaussian, where* f *is the function defined by the operator* $\circ$, *and that (column) vectors* $\mathbf{a}_i = [a_{1i}, a_{2i}, \ldots, a_{pi}]^T$, $i = 1, 2, \ldots, m$, *are linearly independent. Then the model (6) is separable up to distortion* $\kappa_i = f(c_i f^{-1}(S_i) + d_i)$, *where* c_i *and* d_i *are constants,* $i = 1, 2, \ldots, m$.

Proof. The equations (4) and (5) give

$$\mathbf{Y} = \begin{bmatrix} f^{-1}(X_1) \\ f^{-1}(X_2) \\ \vdots \\ f^{-1}(X_p) \end{bmatrix} = \begin{bmatrix} a_{11}f^{-1}(S_1) + a_{12}f^{-1}(S_2) + \cdots + a_{1m}f^{-1}(S_m) \\ a_{21}f^{-1}(S_1) + a_{22}f^{-1}(S_2) + \cdots + a_{2m}f^{-1}(S_m) \\ \vdots \\ a_{p1}f^{-1}(S_1) + a_{p2}f^{-1}(S_2) + \cdots + a_{pm}f^{-1}(S_m) \end{bmatrix} = [\hat{a}_1\ \hat{a}_2\ \cdots\ \hat{a}_m] \begin{bmatrix} f^{-1}(S_1) \\ f^{-1}(S_2) \\ \vdots \\ f^{-1}(S_m) \end{bmatrix} = Af^{-1}(\mathbf{S})$$

which is the linear ICA equation (3), and therefore has the solution of the form $Z_i = c_i f^{-1}(S_{\sigma(i)}) + d_i$, $i = 1, 2, \ldots, m$ and σ permutation, from which we get the theorem by taking the mapping f. □

The previous theorem indicates that the ICA model (6) can be solved (up to the distortions of the above theorem) with the generic algorithm:

1. For each component i, make the transformation $Y_i = f^{-1}(X_i)$.
2. Solve the mixture $\mathbf{Y} = [Y_1, Y_2, \ldots, Y_p]^T$ as linear ICA problem to obtain linearly independent components Z_i, $i = 1, \ldots, p$.
3. Transform each component Z_i with f to obtain the final solution.

There is another consequence of the addition theorem for nonlinear mixing. Namely, if f is any invertible, continuous mapping defined on the entire real axis, a (commutative) group operation can be defined as $\mathcal{F}_2(u, v) = f(f^{-1}(u) + f^{-1}(v))$ on the open set $f(\mathbb{R}) = (\alpha, \beta)$. Therefore, it is seen that in the post-nonlinear model [TJ99],

$$X_i = f_i\left(\sum_{j=1}^{m} a_{ij} Z_j\right), \quad i = 1, \ldots, p, \tag{7}$$

the component mixing operations can be also viewed as a case of addition theorem mixing for variables $Z_i = f_i^{-1}(S_i)$, $i = 1, \ldots, m$. However, it should be noted that in the post-nonlinear mixing model the functions f_i and f_j, $i \neq j$, can be different and moreover they are assumed to be unknown.

4 Examples

As an example, consider the ICA model (1) with nonlinear mixing $\mathcal{F} : \mathbb{R}^3 \to \mathbb{R}^3$ such that each component $\mathcal{F}^{(i)}$ is given by $a_{i1} \star S_1 \circ a_{i2} \star S_2 \circ a_{i3} \star S_3$, where $u \circ v = uv/(u + v)$, $i = 1, 2, 3$. The model has the addition theorem (4) solution $f(x) = 1/x$. Thus the inverse mapping is given by $f^{-1}(x) = 1/x$, and the model can be written explicitly as

$$X_i = \frac{S_1 S_2 S_3}{a_{i3} S_1 S_2 + a_{i2} S_1 S_3 + a_{i1} S_2 S_3}, \quad i = 1, 2, 3. \tag{8}$$

This model has nonlinear distortion given by $s/(c + ds)$.

Suppose each component S_i, $i = 1, 2, 3$, is standard normal distributed. Since the inverse of the normal distribution is not normal, this mixture should be separable. In Matlab simulations, the coefficients were randomly generated, and the linear part of the generic algorithm was solved with the standard JADE algorithm [Car]. Since the distortion is nonlinear, the direct mean square error is not a good quantitative measure of the separation result. However, the distortion of inverses (i.e. $f^{-1}(S_i)$ and $f^{-1}(Y_i)$) is linear. Therefore we measure the Signal-to-Interference-Ratio (SIR(dB)$= -10 \log_{10}$(MSE)) between inverses, which are further normalized to zero mean, unit variance, and the permutation ambiguity is solved. The results are compared with the results obtained by using JADE directly to the mixture, i.e. assuming that the data follows the linear model. The results are given in Figure 1, where also the direct SIR values are given for the reference. The results are averaged over component signal SIR values computed over 1000 Monte Carlo runs. It can be seen that the independent components are reliably found.

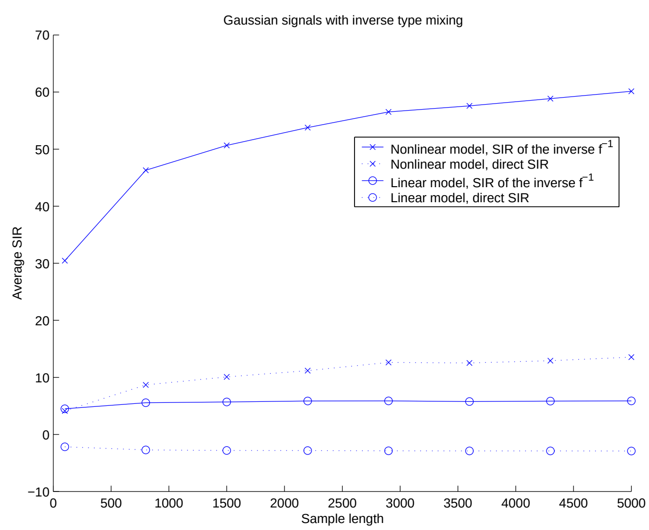

Figure 1: SIR values for different signal lengths in the first example. The mixture is three standard normal variables with the nonlinear $uv/(u + v)$ mixing (8).

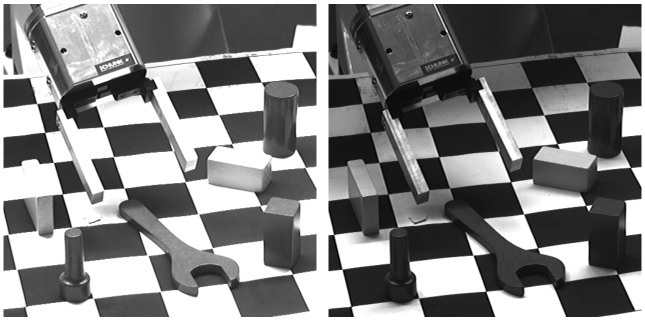

Figure 2: Original images (bt33.tif and bt36.tif)

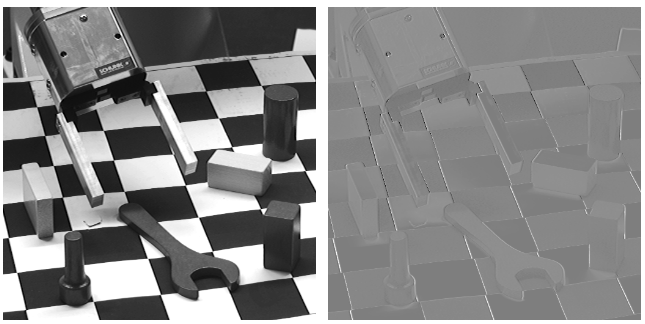

Figure 3: Separation results. The impact of reflectivity has been reduced making reliable image analysis possible. The component on the right is given in log-scale for visualization purposes.

Another example considered is two dimensional mixing, where $u \circ v = uv$, see Table 1. Thus the model can be explicitly written as

$$X_i = S_1^{a_{i1}} S_2^{a_{i2}} \quad i = 1, 2. \tag{9}$$

The basic nature of a gray-level discrete image pixel value $f(x, y)$ at spatial coordinates (x, y) can be modeled [GW92] as $f(x, y) = i(x, y)r(x, y)$, where $i(x, y)$ is the amount of source light incident on the scene being viewed (illumination) and $r(x, y)$ is the amount of light reflected by the objects in the scene (reflection). If object surfaces in the image are specular, image analysis and feature extraction may become unreliable. We used the model (9) to get an enhanced picture of two images taken from the same scene under two different lightning conditions [Pau]. The original images are shown in Figure 2 and the separation results in Figure 3.

5 Conclusion

In this paper we constructed a proof of identifiability to a class of nonlinear instantaneous ICA models. These nonlinearities satisfy so-called addition theorem that covers many nonlinearities of practical interest. An algorithm for separating such nonlinear ICA models was given. The feasibility of the approach was demonstrated using two examples. Especially the image processing example is of high practical interest. The performance and applications of the method will be studied in detail in a forthcoming paper.

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