On the maximum common rate in multiple access channels

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ABSTRACT

Given a multiple access channel, it is well known that the criterion of maximum sum-rate, under the constraint of a given transmitted power for each user, leads to the multiuser water-filling solution. However, maximizing the sum-rate may yield very low (even null) rates for users experiencing the deepest fades. Our goal is to introduce a fairness principle so that all users can transmit at the same rate. The question is then what is the maximum value of such a common rate, under the constraint of a fixed total transmitted power. In this paper, we will show how to solve this problem, providing this maximum common rate and the power allocation and code selection necessary to achieve it. In particular, we will show that in case of transmission over linear time-invariant channels, the solution for the finite block case, using cyclic prefixes, is given by an OFDMA strategy, with a proper power and frequency distribution among the users.

1 System Model and Problem Formulation

The full characterization of a multiple access channel, as far as the maximum reliable user rates are concerned, requires the determination of the capacity region. In case of linear time-invariant channels, additive Gaussian noise, and fixed average power budget for each user, the capacity region for the multiple access channel was initially given in [2]. Even though the capacity region describes the situation completely, nevertheless it is important to single out a few performance parameters to characterize a given channel. For this reason, several works concentrate on the sum-rate, as a single performance parameter. In [2] it was shown that the optimal coding strategy for maximizing the sum of the rates in the multiple access channel, leads to a frequency division multiple access (FDMA) strategy where each frequency is assigned to the user who has the maximum gain over that frequency, after a proper scaling necessary to enforce the desired transmission power to all users. The power allocation within the band assigned to each user is then carried out according to the water-filling algorithm. Quite recently, in [3] an iterative algorithm was given that is able to provide the optimal codes for the multiple access channel, without making any specific assumption about the channel, under the constraint of a fixed average power for each user. In [4] it was proved that in case of block transmission using cyclic prefixes of length at least equal to the channel order, the algorithm of [3] leads to an Orthogonal Frequency Division Multiple Access (OFDMA) strategy, where distinct users get non-overlapping frequency bins (or at most they have only one common frequency bin).

In the algorithms given in [3], [4], the optimization criterion is the maximization of the sum-rate. As such, the algorithms may assign very low (even null) rates to the users experiencing the strongest attenuation. The aim of this paper is to propose a method which avoids the possibility of such an unfair rate distribution. Different from [3], [4], we relax the condition on the power budget, as we put a constraint only on the global available power, and then we find out the set of powers and codes that lead to the same rate for all users. Instead of characterizing the performance of a multiple access channel in terms of sum-rate, we use the *common rate*, so that we are only interested to powers and codes that yield the same rate to all users. Then we optimize the performance of our system by looking for the power distribution and codes selection which yield the maximum common rate, for a given global available average power.

We consider block transmission over frequency-selective time-invariant channels, modeled as FIR filters of maximum order L. We denote by $s_k(n)$ the information symbol stream of the k-th user and with Q the number of simultaneously active users. Each stream is parsed into blocks of M symbols $\boldsymbol{s}_k(n) := [s_k(nM), \ldots, s_k(nM + M - 1)]^T$, with $k = 1, \ldots Q$. To avoid InterBlock Interference (IBI), we introduce, as in any OFDM system, a cyclic prefix (CP) of length L at the beginning of each transmitted block[1]. Adopting linear (redundant) precoding, the n-th block $\bar{\boldsymbol{x}}_k(n) := [x_k(nN), \ldots, x_k(nN+N+L-1)]^T$ transmitted from the k-th user is related to $\boldsymbol{s}_k(n)$ by the relationship $\bar{\boldsymbol{x}}_k(n) = \bar{\boldsymbol{F}}_k \boldsymbol{s}_k(n)$, where $\bar{\boldsymbol{F}}_k$ is an $(N + L) \times M$ full column rank matrix, with $N \geq M$. The first L rows of $\bar{\boldsymbol{F}}_k$ are equal to the last L ones and thus they take into account the CP. At the receiver, the IBI is eliminated by simply discarding the first L samples of the received block and the $N \times 1$ IBI-free received vector $\boldsymbol{y}(n)$ can be written as:

$$\boldsymbol{y}(n) = \sum_{k=1}^{Q} \boldsymbol{H}_k \boldsymbol{F}_k \boldsymbol{s}_k(n) + \boldsymbol{\eta}(n) \tag{1}$$

where $\boldsymbol{F}_k$ is the $N \times M$ coding matrix given by the lower N rows of $\bar{\boldsymbol{F}}_k$; $\boldsymbol{H}_k$, thanks to the insertion of the CP, is

[1] Assuming a quasi-synchronous model, the baseband channel order is $L = \lceil(\tau_d^{max} + \tau_k^{max})/T_c\rceil$, where τ_d^{max} is the maximum delay spread among the channels and τ_k^{max} is the maximum relative delay among the users.

*This work has been supported by the project ROMANTIK funded by the European Community.

an $N \times N$ circulant Toeplitz matrix with entries $H_k(i,j) = h_k((i-j) \mod N)$; $\boldsymbol{\eta}(n)$ is white additive Gaussian noise with covariance matrix $\boldsymbol{R}_\eta \doteq \sigma_n^2 \boldsymbol{I}_N$. We assume also, without any loss of generality, that the information symbols $\boldsymbol{s}_k(n)$ are uncorrelated (any correlation of $\boldsymbol{s}_k(n)$ could in fact be taken into account by including a proper whitening matrix in $\boldsymbol{F}_k$), with covariance matrix $\boldsymbol{R}_{s_k} = \boldsymbol{I}_M$. Symbols transmitted by different users are statistically independent. Under the hypothesis of additive Gaussian noise and the constraint on the average transmitted power, mutual information is maximum when the symbol vectors $\boldsymbol{s}_k(n)$ are Gaussian random variables.

Assuming that all channels $\{\boldsymbol{H}_k\}_{k=1}^Q$ are perfectly known to both transmitters and receivers, the mutual information $\boldsymbol{I}(\boldsymbol{X}_Q;\boldsymbol{y})$ between all the transmitted blocks $\boldsymbol{X}_Q \equiv \{\boldsymbol{x}_1(n), \dots \boldsymbol{x}_Q(n)\}$ and the received vector $\boldsymbol{y}(n)$ is (see, e.g. [1]): $\boldsymbol{I}(\boldsymbol{X}_Q;\boldsymbol{y}) = 1/N \left| \sum_{k=1}^Q \boldsymbol{H}_k \boldsymbol{R}_{x_k} \boldsymbol{H}_k^H + \boldsymbol{R}\eta \right| / |\boldsymbol{R}\eta|$, where the symbol $|\boldsymbol{A}|$ denotes the determinant of $\boldsymbol{A}$. Because of the independence between the symbols transmitted from different users, the maximum sum of achievable users' rates can be computed as the maximum of $\boldsymbol{I}(\boldsymbol{X}_Q;\boldsymbol{y})$. To make evident the role of the power p_k and the code matrix $\boldsymbol{F}_k$ assigned to each user, in the following we use the normalized coding matrices $\boldsymbol{G}_k = \boldsymbol{F}_k/\sqrt{p_k}$, such that $tr\{\boldsymbol{G}\boldsymbol{G}^H\} = 1$ by construction, whereas $tr\{\boldsymbol{F}\boldsymbol{F}^H\} = p_k$. Therefore, the sum-rate maximization problem (SRMP) with unknowns $\{\boldsymbol{G}_k\}_{k=1}^Q$, subject to the power constraint $tr\{\boldsymbol{G}_k\boldsymbol{G}_k^H\} \le 1$, with $k = 1, 2 \dots Q$, can be formulated as follow:

$$\begin{aligned} \{\boldsymbol{G}_k\}_{k=1}^Q &= argmax \log \frac{\left|\sum_{k=1}^Q p_k \boldsymbol{H}_k \boldsymbol{G}_k \boldsymbol{G}_k^H \boldsymbol{H}_k^H + \boldsymbol{R}_\eta\right|}{|\boldsymbol{R}_\eta|} \\ \text{subject to} \quad & tr\{\boldsymbol{G}_k \boldsymbol{G}_k^H\} \le 1, \;\; k = 1 \dots Q. \end{aligned} \tag{2}$$

The solution of the SRMP (2) was given in [3], where it was shown that the optimal coding matrices $\boldsymbol{G}_k$ can be found jointly as the single-user water-filling solutions corresponding to the equivalent channels characterized by the matrix $\boldsymbol{H}_k$ and having as additive noise the sum of the receiver noise plus the interference from all the users (except the k-th one) $\boldsymbol{w}_k(n) \doteq \sum_{j \ne k}^Q \sqrt{p_j} \boldsymbol{H}_j \boldsymbol{G}_j \boldsymbol{s}_j(n) + \boldsymbol{\eta}(n)$. Since the optimal code for each user depends on the codes of all other users, which play the role of interference, the solution can be found only acting *jointly* on all coding matrices. The iterative algorithm able to provide these optimal matrices was given in [3], where it was also proved that the algorithm always converges towards the unique absolute maximum. In case of transmission over time-invariant channels, the solution for each user is the classical water-filling solution corresponding to the case of additive colored noise and it can be expressed as follows. We start with the eigen-decomposition (see also [4]):

$$\boldsymbol{H}_k^H \boldsymbol{R}_{w_k}^{-1} \boldsymbol{H}_k = (\boldsymbol{V}_k \; \widetilde{\boldsymbol{V}}_k) \begin{pmatrix} \boldsymbol{\Lambda}_k & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} (\boldsymbol{V}_k \; \widetilde{\boldsymbol{V}}_k)^H, \tag{3}$$

where $\boldsymbol{V}_k$, $\widetilde{\boldsymbol{V}}_k$ are respectively $N \times r$ and $N \times (N-r)$ paraunitary matrices, with $r = rank(\boldsymbol{H}_k^H \boldsymbol{R}_{w_k}^{-1} \boldsymbol{H}_k) \le N$, and $\boldsymbol{\Lambda}_k$ is an $r \times r$ diagonal matrix with entries $\lambda_k(i,i)$. The optimal code matrices $\boldsymbol{F}_k$ maximizing the sum-rate are (see e.g. [1]):

$$\boldsymbol{F}_k = \boldsymbol{V}_k \boldsymbol{\Phi}_k, \tag{4}$$

where $\boldsymbol{\Phi}_k$ is a diagonal matrix whose entries are such that

$$\boldsymbol{\Phi}_k^2(i,i) = max\left(\frac{P_k + \sum_{j=1}^{\bar{M}_k} 1/\lambda_k(j,j)}{N} - \frac{1}{\lambda_k(ii)}, 0\right) \tag{5}$$

and $\bar{M}_k$ is such that $\sum_{j=1}^{\bar{M}_k} \boldsymbol{\Phi}_k^2(j,j) = p_k$. In [4] it is shown that the orthonormal matrix $\overline{\boldsymbol{V}}_k = (\boldsymbol{V}_k \widetilde{\boldsymbol{V}}_k)$ is $\boldsymbol{W}$ $\forall k \in [1, \dots Q]$, with $\{\boldsymbol{W}\}_{kl} = \exp(j2\pi kl/N)/\sqrt{N}$ and the precoding matrix $\boldsymbol{F}_k$ in (4) assumes the form $\boldsymbol{F}_k = \boldsymbol{W}\boldsymbol{P}_k\overline{\boldsymbol{\Phi}}_k$, where $\boldsymbol{P}_k$ is a permutation matrix such that $\boldsymbol{W}\boldsymbol{P}_k$ contains the eigenvectors corresponding to the eigenvalues of the $N \times N$ diagonal matrix $\overline{\boldsymbol{\Phi}}_k = diag\left([\boldsymbol{\Phi}_k]_{11}, \;\dots, [\boldsymbol{\Phi}_k]_{rr}, 0, \dots, 0\right)$ arranged in decreasing order. Therefore the transmission strategy $\{\boldsymbol{F}_k\}_{k=1}^Q$ maximizing the sum-rate is the multi-carrier one, with proper power loading across the sub-carriers.

2 Water-Filling with Fairness Constraint

The iterative multi-user water-filling algorithm (IMUWFA) proposed in [3] provides the codes yielding the maximum sum-rate, after a few iterations. However, this solution might give rise to a completely unfair distribution of the rates across the active users. In particular, the users with the strongest channel attenuation could get a very low, or even null, rate. To establish a fairness principle while, at the same time, looking for a single parameter characterizing the multiple access channel, we formulate the optimization problem differently. We relax the constraint about the available power, as we impose only that the global available power (the sum of the user powers) be less than a given value and we look for the distribution of powers and codes which lead to the *same rate* for all active users. This common rate, as opposed to the sum-rate, is thus our performance parameter. Our goal is then to find out the power and codes distribution which maximize the common rate. Different from [3], [4], our algorithm exploits the flexibility in assigning the power to each user, within the only constraint of a maximum global available power, to enforce the same rate to all users and thus our optimization involves the selection of powers and codes of each user *jointly*. Stated in mathematical terms, denoting with R_c the common rate, which is a function of the power p_k and the normalized coding matrices $\boldsymbol{G}_k$ assigned to each user, we look for the powers and coding matrices such that

$$\begin{aligned} &\{p_1^*, \cdots, p_Q^*; \boldsymbol{G}_1^*, \cdots, \boldsymbol{G}_Q^*\} = argmax\{R_c\} \\ &\text{subject to} \quad \sum_{k=1}^Q p_k^* tr\{\boldsymbol{G}_k \boldsymbol{G}_k^H\} \le P_{tot}, \end{aligned} \tag{6}$$

where the stars indicate the optimal solution. Clearly this problem is nonlinear with a (possibly) high number of unknowns. Therefore, its numerical solution can be very complicated. However, there is a crucial remark which simplifies the problem considerably. Among all possible power distributions, satisfying the global power constraint, there is certainly a non-empty set of power distributions such that the rates assigned to different users are all equal to each other. Within this set, the coding matrices yielding the maximum common rate must coincide with the coding matrices giving the maximum sum-rate. In fact, when the rates are all equal to each other, maximizing the common rate R_c is clearly equivalent to maximizing QR_c, that is the sum-rate. This remark, although trivial, yields a great simplification in the solution of our problem, because it implies that if we know the optimal powers, we can derive the optimal codes using for example the IMUWFA ([3]. On the other hand, we know that, for any given power distribution among the users, the rates obtained with the IMUWFA are determined uniquely. Hence, given any iterative method acting on the power distribution in order to reduce the difference between the user rates, if it converges, we are sure that it converges towards the unique solution. Our algorithm proceeds then through

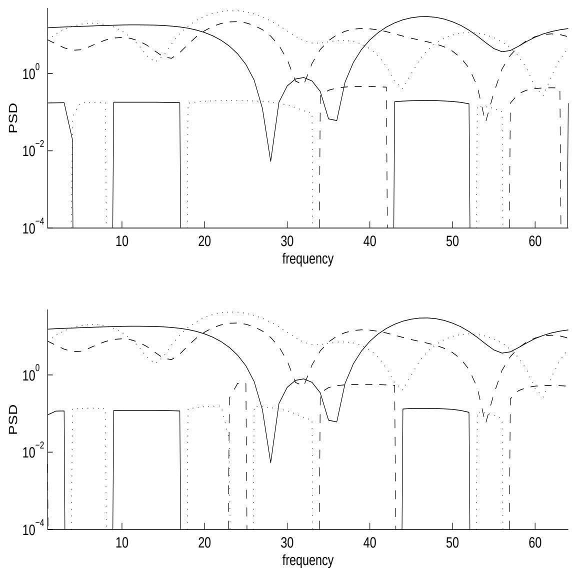
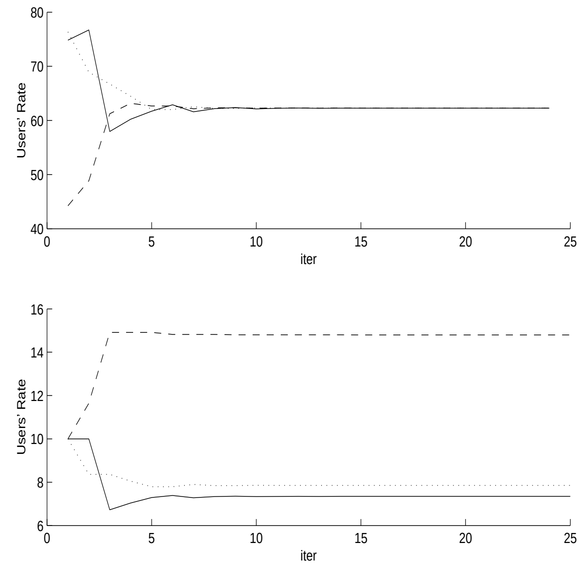


Figure 1: Channel transfer functions (upper curves) and optimal user PSD's without fairness (top figure) and with fairness constraint (bottom figure).



b)

Figure 2: Rates (upper curves) and powers (lower curves) of each users, vs. iteration index.

the following steps.

Algorithm

- Start with any power distribution $\boldsymbol{p}(1) \equiv \{p_1(1), \ldots, p_Q(1)\}$ such that $\sum_{k=1}^{Q} p_k(1) = P_{tot}$ and set $n = 1$;
- Repeat

  i) Compute the code matrices $\boldsymbol{G}_k(n)$, for $k = 1, \ldots, Q$ yielding the maximum sum-rate, corresponding to the set of powers $\boldsymbol{p}(n)$, using the IMUWFA [3];

  ii) Evaluate the rates $R_k(n)$ for each user and compute the mean rate $\bar{R}(n) := \sum_{k=1}^{Q} R_k(n)/Q$ and the dispersion rate $\sigma_R(n) := \sqrt{\sum_{k=1}^{Q} (R_k(n) - \bar{R}(n))^2/Q}$;

  iii) associate to each user its own rate distance with respect to the average rate $\Delta_k(n) := R_k(n) - \bar{R}(n)$ and sort the users in decreasing order according to the distribution of $|\Delta_k(n)|$;

  iv) update the powers vector in order to reduce the rate dispersion:

$$p_k(n+1) = p_k(n) - \mu \frac{R_k(n) - \bar{R}(n)}{\bar{R}(n)}, k = 1, \ldots, Q-1;$$

$$p_Q(n+1) = P_{tot} - \sum_{k=1}^{Q-1} p_k(n+1);$$

  v) set $n = n + 1$;

- until $|R_k(n) - \bar{R}(n)| < \epsilon, \forall k$.

In step iv) we update the powers using an updating term which is directly proportional, for each user, to the distance between the rate of that user and the average rate. The power of the Q-th user is updated in order to enforce the global power constraint. The algorithm stops when the difference between the rate of any user from the average rate falls below a given threshold ϵ. In the next section, we will show the performance of our algorithm using different power updating strategies and we will show that all strategies lead to the same final results, as the only difference is convergence time. It is important to emphasize the role of step i), where the code matrices are computed, for any power distribution, in order to maximize the sum-rate. This step insures that if the algorithm converges, the final result is not only common rate, but maximum common rate.

3 Performance and Conclusions

We have simulated our algorithm using the following system configuration. The number of active users is $Q = 3$; the length of the transmitted blocks is $N = 64$ and the maximum number M of information symbols in each coded block is $M = N$; the channels are simulated as FIR filters of order $L = 10$, whose taps are iid complex Gaussian random variables with zero mean and unit variance; the additive noise $\boldsymbol{\eta}(n)$ is assumed to be drawn from a complex white Gaussian random process with zero mean and unitary variance for each component. In Fig.1 we compare our algorithm (bottom figure) with IMUWFA (top figure). In each plot we report the square modulus of the channels transfer functions (upper curves) and the optimal power spectral densities (PSD) (lower curves). The total power is the same for both algorithms and IMUWFA assigns the same power to all users. Interestingly, in both cases the solution is an OFDMA scheme, where different users get non-overlapping frequency bands, and within each band the power distribution goes according to the water-filling principle. We can observe that our algorithm gives some extra band to the most penalized user (the one with dashed line), with respect to IMUWFA. To quantify the rate dispersion as a function of the iteration index, in Fig.2 we report the rates and powers of each user as a function of the iteration index, for the same channel configuration as in Fig.1. At the beginning of the iterations, we assign the same power to all users and we can clearly see that one user gets less than 60 percent of the rates assigned to the other users. Then, after a few iterations, the powers assigned to each user changes in order to give the same rate to all users. To analyze the convergence properties of

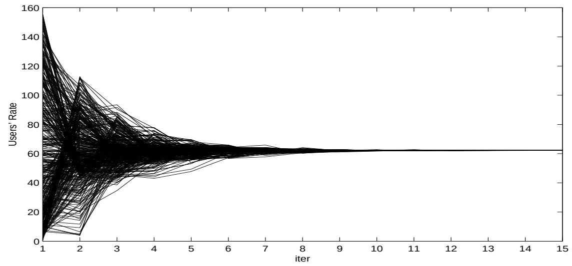


Figure 3: Rate of three users vs. iteration index, using random power initialization.

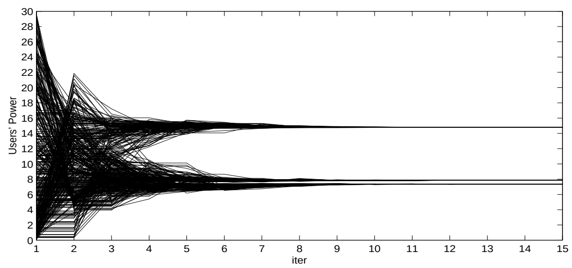


Figure 4: Powers of three users vs. iteration index, using random power initialization.

our algorithm, in Fig.3 and Fig.4 we plot the rates (upper curves) and the powers (lower curves) of three users as a function of the iterations, obtained running our algorithm and using 5000 independent random initial power distributions $\boldsymbol{p}(0)$. From these figures we can clearly observe that our algorithm converges always to the *same* common rate R_c, independently of the initial power allocation, and also the final powers assigned to the three users are always the same. Finally, in Fig. 5 we compare three different strategies for updating the power distribution in order to enforce the same rate for all users: i) the method described in the above algorithm (dashed and dotted line); ii) a method updating the powers in order to maximize the geometric mean of the rates, for a given sum-rate (solid line) - it is well known in fact that the maximum of the geometric mean of a set of unknowns, under the constraint of a given sum, is achieved when all the unknowns assume the same value; iii) a method updating the powers by subtracting the power from the user with the highest $\Delta_k(n)$ in order to make its rate equal to the average rate, at that iteration, and adding this same power to the user with the lowest rate, to respect to power constraint. Clearly, the ordering of the users in step iii) plays a fundamental role in this updating strategies. From Fig. 5, we can check that indeed the only difference between the different methods is convergence time, but not the final values, both in terms of powers and rates. In conclusion, the use of the common rate as a unique performance parameter avoids the possibility of unfair rate distribution among users. In practice, different users might ask for different rate service, in which case, the powers assigned to each user should vary accordingly. Nonetheless, the maximum common rate is a useful parameter to characterize a given network configuration, irrespective of what the users are going to ask. In [1] we have generalized the algorithm presented in this paper to systems using multiple antenna transceivers.

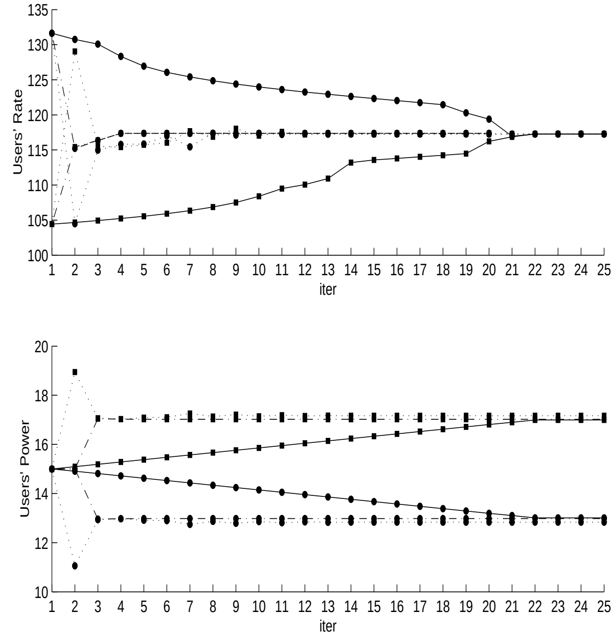


Figure 5: Comparison of three different power updating strategies.

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