A Robust Array Signal Processing Maximum Likelihood Estimator Based on Sub-Gaussian Signals

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ABSTRACT

In this work we investigate an alternative to the stochastic Gaussian *Maximum Likelihood* (ML) method that deals with sub-Gaussian signals. The proposed system is one where the sources are stochastic and Gaussian and the transfer medium is varying in a highly impulsive manner, introducing the sub-Gaussian nature at the receiver. Alternatively, the impulsive transformation to the signals can be viewed as part of the source model, creating a multivariate source signal whose components can not be independent, and is of impulsiveness equal to the one of the Cauchy distribution.

The Lévy α-stable distribution, of characteristic exponent 0.5 and index of symmetry 1, is used together with the multivariate Gaussian density to model the signal, and the resulting probability density function is derived. Based on this density, the stochastic ML estimator is formulated. A separable solution of the estimator is given, and simulations demonstrating the performance gains relative to the Gaussian-based ML estimator are provided.

1 Introduction

The Gaussian distribution has traditionally been the most widely accepted distribution and used, as a rule, as a realistic model for various kinds of noise. In recent years however, there has been a tremendous interest in the class of α-stable distributions, which are a generalization of the Gaussian distribution, but are able to model a wider range of phenomena and can be of a more impulsive nature. In fact, the Gaussian is the least impulsive α-stable distribution, while other widely known distributions of the α-stable class are the Cauchy and the Lévy.

In 1991, Cambanis, Samorodnitsky, and Taqqu [1] gave a review of α-stable processes from a statistical point of view. Several other statisticians have provided valuable work in the theory of α-stable distributions. In 1993, Nikias and Shao gave an introductory review of α-stable distributions from a statistical signal processing viewpoint that was followed by a book from the same authors in 1995 [2].

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Alpha-stable distributions have been used to model diverse phenomena such as random fluctuations of gravitational fields, economic market indices, and radar clutter. These authors have presented in previous work [3] the appropriateness of the α-stable distributions for modeling noises encountered in audio environments and presented a time delay estimation method for localization of speech sources. Tsakalides and Nikias [4, 5] gave *Maximum Likelihood* (ML) and *Multiple Signal Classification* (MUSIC) based localization algorithms for uncorrelated, impulsive signals. In this paper we will present a ML algorithm for signals that are not independent and impulsive in nature.

1.1 Sub-Gaussian Random Variables

Sub-Gaussian distributions are a special case of α-stable random processes. A Sub-Gaussian random vector $\boldsymbol{X}$ can be defined as a random vector with characteristic function of the form

$$\varphi(\boldsymbol{u}) = \exp\left(-\frac{1}{2}\left[\boldsymbol{u}^T\,\mathbf{R}\,\boldsymbol{u}\right]^{\alpha/2}\right) \tag{1}$$

where $\mathbf{R}$ is a positive-definite matrix, and the characteristic exponent satisfies $1 < \alpha \leq 2$.

Sub-Gaussian processes are variance mixtures of Gaussian processes [6]. Specifically, $\boldsymbol{X}(t)$ is sub-Gaussian with parameter α if $S(t)$ is a positive stable process with characteristic exponent $\alpha/2$ (*i.e.,* S is $\alpha/2$-stable random variable completely skewed to the right) and dispersion $\cos\left(\frac{\pi\alpha}{4}\right)^2$, and $\boldsymbol{Y}(t)$ is a multivariate Gaussian process independent of S, and:

$$\boldsymbol{X}(t) = S(t)^{1/2}\boldsymbol{Y}(t) \tag{2}$$

2 Maximum Likelihood Estimation

The transmitted signals in this case are assumed to be stochastic, and as such, the parameters of interest will be their statistics and *Directions-of-Arrival* (DOA's). Despite the wide variety of optimization criteria used for parameter estimation, the optimal detector is characterized by a single result: the Maximum Likelihood ratio test, which was also one of the first methods to be applied in the area of array signal processing [7]. In this paper, we deal exclusively with *Stochastic ML* estimation where the signals are assumed to be random rather than of a deterministic nature.

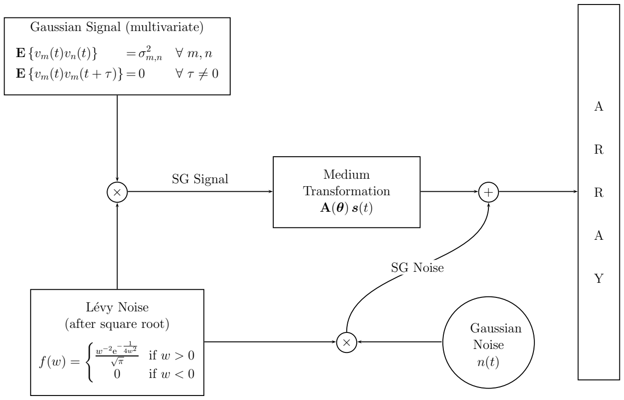

Figure 1: A multivariate Gaussian signal, corrupted by multiplicative Lévy noise, is then transformed through a set of delays to the receiving end of the array. The additive noise can be generated from the same Lévy process to make it jointly sub-Gaussian with the signal.

We assume a scenario under which there are κ sources received by an array of ρ sensors. The transfer function each signal undergoes while traveling to the array can be modeled as an attenuation and a delay. The attenuation will be considered the same at all sensors under the assumption that the sources are in the far field of the array. These transfer functions are

$$a_{r,k} = \mathrm{e}^{-\mathrm{i}\omega\tau_{r,k}}, \quad r = 1\ldots\rho \quad \text{and} \quad k = 1\ldots\kappa \qquad (3)$$

where $\tau_{r,k}$ is the delay of the signal (of source k) received at sensor r relative to the first sensor.

We assume the sources to be in the far field and hence, $\tau_{r,k} = \tau_r(\theta_k)$, and it is also clear that if we are dealing with a linear array $\tau_{r,k} = (r-1)\cdot\tau_1(\theta_k)$.

We denote the vector of the medium transformations for source k by $\boldsymbol{a}_k = [a_{1,k} \quad a_{2,k} \quad \ldots \quad a_{\rho,k} \quad]^T$.

The array's input at a single sensor r is

$$x_r(t) = \sum_{k=1}^{\kappa} \boldsymbol{a}_{r,k} \cdot s_k(t) + n_r(t) \qquad (4)$$

and therefore, the array's input vector is

$$\boldsymbol{x}(t) = \mathbf{A} \cdot \boldsymbol{s}(t) + \boldsymbol{n}(t) \qquad (5)$$

3 Signal Model for ML Estimation

An alternative to modeling the signal as Cauchy distributed, which was pursued by [4], is using a Sub-Gaussian signal of equal impulsiveness. For this purpose we can use a distribution of impulsiveness $\alpha = 0.5$, which is completely skewed to the positive axis together with a multivariate Gaussian density. Fortunately, there is one distribution with a closed form expression, the Lévy distribution, which satisfies exactly these properties (also referred to as a Paretto type 5 distribution with an index of symmetry $\beta = 1$ and characteristic exponent $\alpha = 0.5$). Fig. 1 gives a top level description of the problem, source, and noise signals:

The Gaussian density is:

$$f(\mathbf{V}) = \prod_{t=t_1}^{t_M} \frac{1}{\pi^\rho\,|\mathbf{R}|} \exp\left(-\boldsymbol{v}^\dagger(t)\mathbf{R}^{-1}\boldsymbol{v}(t)\right) \qquad (6)$$

where

$$\mathbf{v} = \boldsymbol{v}(t_1),\, \boldsymbol{v}(t_2), \ldots,\, \boldsymbol{v}(t_M) \qquad (7)$$

and the Lévy distribution [8] is given by:

$$f(u) = \begin{cases} \frac{u^{-\frac{3}{2}}\, \mathrm{e}^{-\frac{1}{4u}}}{2\sqrt{\pi}} & \text{if } u > 0 \\ 0 & \text{if } u < 0 \end{cases} \qquad (8)$$

So from eq. (2), the signal $\boldsymbol{s} = [s_1 \ldots s_\kappa]^{\mathrm{T}}$ is:

$$s_k(t) = u_k(t)^{\frac{1}{2}} \cdot \boldsymbol{v}_k(t) = w_k(t) \cdot \boldsymbol{v}_k(t) \qquad (9)$$

We can show [9] that the distribution can be given by:

$$f(\boldsymbol{s}) = \frac{1}{2\sqrt{\pi}\,\pi^\kappa\,|\boldsymbol{\Sigma}|} \cdot \left[{}^1\!/_4 + \boldsymbol{s}^\dagger(t)\boldsymbol{\Sigma}^{-1}\boldsymbol{s}(t)\right]^{-1} \qquad (10)$$

Note that if the Gaussian random variable was one dimensional and real, then under the choice of $\sigma = \sqrt{2}$ the sub-Gaussian r.v. would revert to the Cauchy as expected.

$$f(\boldsymbol{s}) = \frac{1}{2\sqrt{2\pi}\sigma} \cdot \left[{}^1\!/_4 + \frac{s^2}{2\sigma^2}\right]^{-1}$$

4 Maximum Likelihood Estimator

Now the signal $\boldsymbol{x} = [x_1 \ldots x_\rho]^{\mathrm{T}}$ is of the form:

$$\boldsymbol{x}_r(t) = y(t)^{1/2} \cdot \boldsymbol{z}_r(t) \qquad (11)$$

where again, as in the transmitted signal case, the received signal is sub-Gaussian

It is therefore straightforward to show that the received signal's $\boldsymbol{z}$ statistics will be relating to those of the transmitted signal $\boldsymbol{v}$ by[1]:

$$\mathbf{R} = \mathbf{A}\,\boldsymbol{\Sigma}_v\,\mathbf{A}^\dagger + \sigma_n^2 \mathbf{I}_\rho \qquad (12)$$

Therefore, the maximum likelihood estimator is

$$[\hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{\theta}}] = \arg\max_{\hat{\boldsymbol{\Sigma}},\hat{\boldsymbol{\theta}}} \prod_{t=t_1}^{t_M} \frac{{}^1\!/_2}{\sqrt{\pi}\,\pi^\rho\,|\mathbf{R}|} \cdot \left[\boldsymbol{x}^\dagger(t)\mathbf{R}^{-1}\boldsymbol{x}(t) + {}^1\!/_4\right]^{-1}$$

To simplify, take the $\log_{\mathrm{e}}$

$$[\hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{\theta}}] = \arg\min_{\hat{\boldsymbol{\Sigma}},\hat{\boldsymbol{\theta}}} \sum_{t=t_1}^{t_M} \Big\{ \log_{\mathrm{e}} |\mathbf{R}| + \log_{\mathrm{e}} \left[\boldsymbol{x}^\dagger(t)\mathbf{R}^{-1}\boldsymbol{x}(t) + {}^1\!/_4\right] \Big\} \qquad (13)$$

[1]With the additional assumption that the noise is a sub-Gaussian process produced by the same Lévy sequence, but not necessarily of the same dispersion. A scalar gain is already incorporated in the medium transformation $\mathbf{A}$ that can modify the dispersion of the Lévy process.

5 Maximum Likelihood: A Separable Solution

5.1 Estimating the Statistics

We proceed in this case to reach an alternative minimization function to reduce the search space. To do so, we follow the same procedure as in [10] (derivations in [9]) where the ML function is minimized with respect to the signal statistics, assuming known DOA:

$$\boldsymbol{\Sigma}_{\mathrm{ML}} = \sum_{t=t_1}^{t_M} \left[\mathbf{A}^{-} \left(\frac{\boldsymbol{x}\boldsymbol{x}^{\dagger}}{\mathrm{Tr}\left[\mathbf{R}^{-1}\boldsymbol{x}\boldsymbol{x}^{\dagger}\right] + {}^{1}/_{4}} - \sigma_n^2 \right) \mathbf{A}^{-\dagger} \right] \tag{14}$$

where $\mathbf{A}^{-} = \left(\mathbf{A}^{\dagger}\mathbf{A}\right)^{-1}\mathbf{A}^{\dagger}$ and $\mathbf{R}^{-1}$ as defined in eq. 15. The solution of eq. 14 can be easily found using the numerical iteration method.

$$\mathbf{R}^{-1} = \frac{1}{\sigma_n^2}\left\{ \mathbf{I} - \mathbf{A}\left(\boldsymbol{\Sigma}\,\mathbf{A}^{\dagger}\,\mathbf{A} + \sigma_n^2\mathbf{I}\right)^{-1}\boldsymbol{\Sigma}\,\mathbf{A}^{\dagger} \right\} \tag{15}$$

An initial estimate of $\mathbf{R}$ can be used as an initial guess and can be found from the data, using a covariation measure. Simulations will not be provided in this paper due to length constraints, however the reader can refer to [9].

The noise variance σ_n^2 can also be found, assuming we know the number of sources and sensors, from the $\rho - \kappa$ smallest eigenvalues of $\mathbf{R}$.

5.2 DOA Estimation

The above sub-section assumes that the DOA vector is known, an issue we investigate here. Using a pseudo-ML approach, we can express the modified ML function irrespective of the statistics $\mathbf{R}$ as

$$\hat{\boldsymbol{\theta}} = \arg\min_{\hat{\boldsymbol{\theta}}} \sum_{t=t_1}^{t_M} \left\{ \log_{\mathrm{e}}\left[\boldsymbol{x}^{\dagger}(t)\mathbf{R}^{-1}\boldsymbol{x}(t) + {}^{1}/_{4} \right] \right\} \tag{16}$$

where $\boldsymbol{\Sigma}$ can be substituted with any valid statistics (identity matrix for instance). A search algorithm can be used to find the solution of the above equation.

6 Simulations – DOA Estimation

Several sets of simulations need to be performed to test the validity of the algorithm. In the following tests, $\boldsymbol{\Sigma} = \mathbf{I}$ is assumed to hold although the test matrix had a random correlation structure, but always with diagonal elements of dispersion equal to 1.

In all cases, the impulsiveness was kept constant ($\alpha = 1$ for cases 1 & 4 and $\alpha = 2$ for 2 & 3 as described below). The Generalized Signal-to-Noise Ratio used below is defined as:

$$\mathrm{GSNR} = 10\log_{10}\left(\frac{\gamma_s}{\gamma_n}\right) = -10\log_{10}\left(\gamma_n\right) \tag{17}$$

Fig. 2 shows the mean squared error and the probability of localization for the conditions described in Fig. 1. Four cases were simulated, and in each case the noise followed the same assumptions as the signal:

1. Exactly as per the derivation assumptions (Fig. 2a): Received signal is sub-Gaussian, created from a Multivariate Gaussian and a univariate Lévy (can be viewed as Lévy energy fluctuation). Received signal impulsiveness is $\alpha = 1$.

2. The signal is a Multivariate Gaussian (Fig. 2b) and is created from a Multivariate Gaussian ($\boldsymbol{v}$) and a univariate Gaussian (w). Received signal impulsiveness is $\alpha = 2$.

3. The signal is a Multivariate Gaussian (Fig. 2c) and it undergoes *no* energy fluctuation ($w = 1$, $\boldsymbol{v} = \boldsymbol{s}$). This conforms to the assumptions of the well known Gaussian based ML. Clearly the received signal impulsiveness is $\alpha = 2$.

4. Finally the received signal is sub-Gaussian (Fig. 2d), created from a Multivariate Gaussian ($\boldsymbol{v}$) and a *Multivariate* Lévy ($\boldsymbol{w}$). In this case, the signals can be viewed as simply Cauchy or as Gaussian with a different Lévy energy fluctuation for each source. Received signal impulsiveness is $\alpha = 1$.

As expected, the sub-Gaussian ML method performs better when the derivation assumptions hold (Fig. 2a). Likewise, when the signal is a multivariate Gaussian, the Gaussian ML algorithm performs better (Fig. 2c).

In the cases that neither assumption holds however, we can see how more robust the sub-Gaussian ML method is. When the signal follows (2) (Fig. 2b), the sub-Gaussian ML performs slightly better than the Gaussian ML. The real benefit of the proposed ML method can however be observed when the signals are impulsive due to random multiplicative noise, independent from one source to the next (Fig. 2d).

7 Conclusions

We have presented an alternative to the multivariate Gaussian for modeling array signal processing sources, and we have derived a ML estimator and a separable solution for the assumed signal conditions.

The suggested model is robust and can model a variety of different phenomena such as sources undergoing the same fluctuation over time, or signals traveling through a rapid varying medium. It can also allow for sparse measurements of certain events in which the measurement conditions might change significantly, but uniformly for the whole array.

The proposed algorithm has also demonstrated significant robustness in localization when tested against different than the ideal conditions, such as those of random gains at each source sample, which could be the model of a non uniform and varying transport medium.

References

[1] Stamatis Cambanis, Gennady Samorodnitsky, and Murad S. Taqqu, Eds., *Stable Processes and Related Topics*, Progress in probability. Birkhäuser, Boston, 1991, Selection of papers from the Workshop on Stable Processes and Related Topics held at Cornell University.

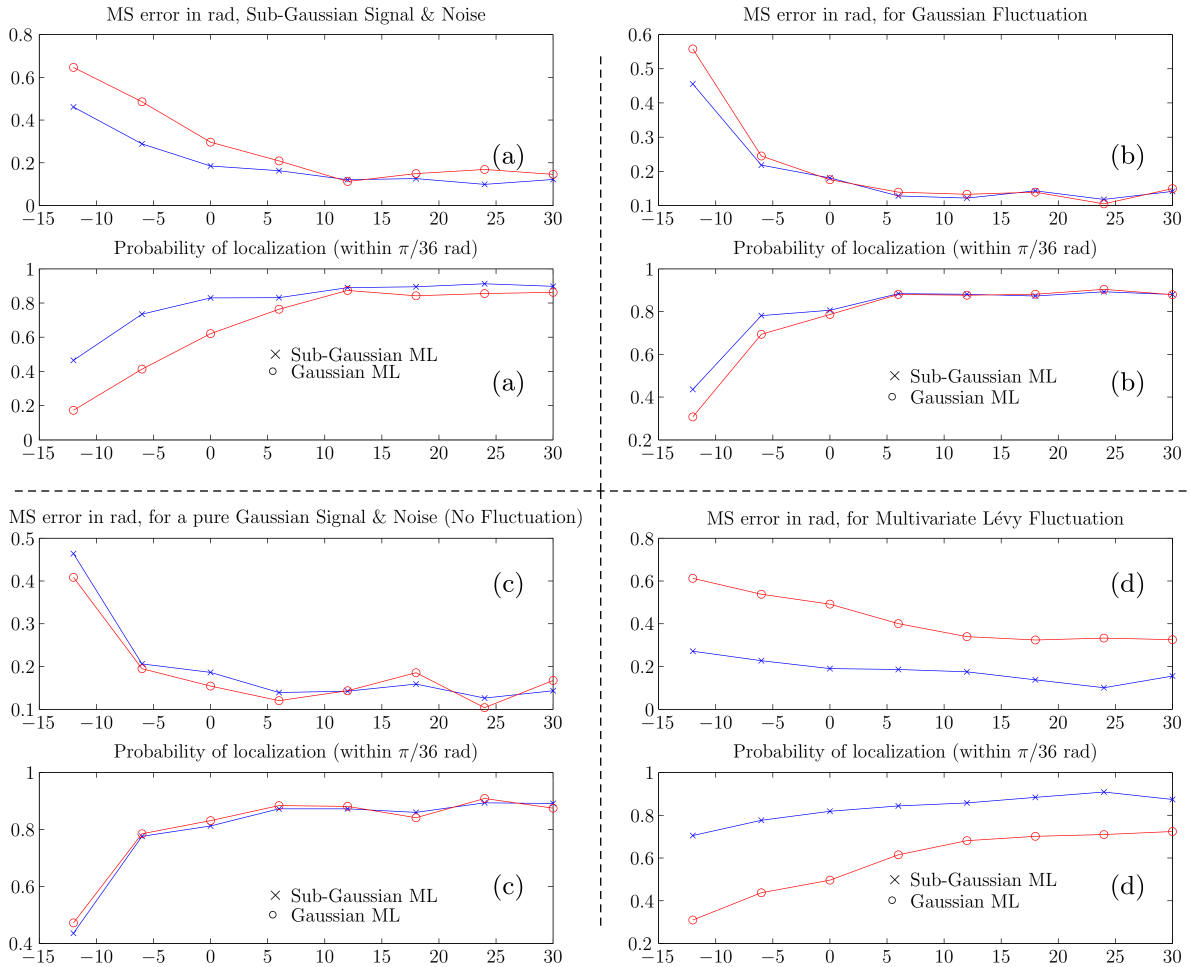

Fig. 2 Localization simulations for different noise conditions as described in text.

[2] C. L. Nikias and M. Shao, *Signal Processing with Alpha-Stable Distributions and Applications*, John Wiley and Sons, New York, 1995.

[3] P. G. Georgiou, P. Tsakalides, and C. Kyriakakis, "Alpha-stable modeling of noise and robust time-delay estimation in the presence of impulsive noise," *IEEE Transactions on Multimedia*, vol. 1, no. 3, pp. 291–301, September 1999.

[4] P. Tsakalides and C. L. Nikias, "Maximum likelihood localization of sources in noise modeled as a stable process," *IEEE Transactions on Signal Processing*, vol. 43, no. 11, pp. 2700–2713, November 1995.

[5] P. Tsakalides and C. L. Nikias, "The robust covariation based music (ROC-MUSIC) algorithm for bearing in impulsive noise environments," *IEEE Transactions on Signal Processing*, vol. 44, no. 7, pp. 1623–1633, July 1996.

[6] S. Cambanis and G. Miller, "Linear problems in p^{th} order and stable processes," *SIAM Journal on Applied Mathematics*, vol. 41, no. 1, pp. 43–69, August 1981.

[7] Don H. Johnson and Dan E. Dudgeon, *Array signal processing: concepts and techniques*, Signal Processing Series. Prentice-Hall, Englewood Cliffs, NJ, 1993.

[8] V. M. Zolotarev, *One-dimensional stable distributions*, vol. 65 of *Translations of mathematical monographs*, American Mathematical Society, Provindence, R.I., 1986.

[9] P. G. Georgiou and C. Kyriakakis, "Maximum likelihood parameter estimation under impulsive conditions. a sub-Gaussian signal approach," Submitted: IEEE Trans. Signal Processing.

[10] A. G. Jaffer, "Maximum likelihood direction finding of stochastic sources: a separable solution," in *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing.*, 1998, vol. 5, pp. 2893–2896.