COMPREHENSIVE EVALUATION OF THEORETICAL APPROXIMATIONS FOR SPECTRAL QUANTIZATION PERFORMANCE

William R. Gardner,[1,2] *Anand D. Subramaniam*[2]*, and Bhaskar D. Rao*[2]

[1]QUALCOMM Inc.
5775 Morehouse Drive
San Diego, CA 92121

[2]Department of ECE
University of California, San Diego
La Jolla, CA 92093-0407

ABSTRACT

In this work, recently derived theoretical approximations for high rate vector quantization (VQ) are reformulated to cleanly separate the effects of the VQ codevector density and the local Voronoi region shapes on overall VQ performance. Numerical evaluation of the resulting theoretical expressions is performed, which allows comparison of the relative importance of codevector density and Voronoi region shapes for a variety of different conditions. In particular, results are presented which compare root mean squared (RMS) vs. mean squared (MS) optimal quantizers, full band vs. partial band log spectral distortion (LSD) quantizers, LPC vs. K vs. LAR vs. ASIN vs. LSP vs. cepstral coefficients, 0th order vs. 1st order recursion, and optimal vs. weighted mean squared error (WMSE) vs. mean squared error (MSE) quantizers.

1. INTRODUCTION

Within the last several years, high rate vector quantization theory has been applied to the fixed rate LPC VQ problem. In [1], high rate expressions for MS full band LSD of various LPC parametric representations and for optimal, WMSE, and MSE-based quantizers were derived. In [2, 3], high rate expressions were derived with more rigor, extended (in the optimal case) for arbitrary powers of the desired distortion measure, and extended to include variable rate quantizers. In [4], the high rate expressions for optimal recursive quantization were derived.

In [1], numerical evaluation of the high rate expressions was limited due to the lack of estimation procedures for the high dimensional LPC vector probability density function. In [4, 5, 6, 7], density estimation procedures utilizing mixtures of Gaussian components were used to estimate the density function of the LPC vector source. In these techniques, the density function is represented as

$$f(\boldsymbol{a}) = \sum_i \alpha_i N(\boldsymbol{\mu_i}, \boldsymbol{\Sigma_i})$$

where $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a Gaussian density function with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and where α_i are weighting factors which sum to 1. Given a database of input training vectors, the factors α_i, $\boldsymbol{\mu_i}$ and $\boldsymbol{\Sigma_i}$ can be iteratively chosen to maximize the log-likelihood of the training database using the expectation maximization (EM) algorithm. In [6], the knowledge of the bounded region of support of the stable LPC vectors was included in the EM algorithm to improve the selection of the parameters. These density functions allowed evaluations of the theoretical expressions for high rate performance. In [4], numerical expressions for the theoretical results for several cases were presented, in particular for partial band, RMS LSD as applied to quantization of cepstral coefficients. The existence of high rate expressions and approximate density functions now allows the numerical evaluation of high rate VQ performance for many different cases, allowing some approximate comparisons of different quantization techniques.

This paper presents a reformulation of the high rate theory of vector quantization which cleanly separates the effects of the VQ codevector density and the local Voronoi region shapes on overall VQ performance. Numerical evaluation of these expressions is performed, which allows comparison of the relative importance of codevector density and Voronoi region shapes for a variety of different conditions. In particular, the results allow for approximate comparisons of different quantization techniques utilizing root-mean-squared (RMS) vs. mean-squared (MS) LSD, full band vs. partial band log-spectral-distortion LSD, LPC vs. K vs. LAR vs. ASIN vs. LSP vs. cepstral coefficients, 0th order vs. 1st order recursion, optimal vs. weighted-mean-squared-error (WMSE) vs. mean-squared-error (MSE) quantizers, and different schemes which may result in optimal or suboptimal VQ densities and/or optimal or suboptimal VQ Voronoi region shapes.

2. REFORMULATED HIGH RATE VQ THEORY

This section provides a reformulation of high rate results for high rate VQ quantization schemes, separating the effects of VQ codevector density and local Voronoi region shapes. Original derivations of the various results can be found in [1, 2, 3, 4, 5, 6].

Let $\mathbf{x}$ be an n dimensional random vector in $D \subset \mathcal{R}^n$ with probability density function $f(\boldsymbol{x}) = f(x_1, x_2, ..., x_n)$. A B bit fixed rate vector quantizer is composed of a quantization function $Q(\boldsymbol{x})$ which maps $\mathbf{x}$ to an output vector $\bar{\boldsymbol{x}}$ in a set of 2^B output vectors in $\bar{\boldsymbol{X}} = \{\bar{\boldsymbol{x}}_1, ..., \bar{\boldsymbol{x}}_{2^B}\}$. Let $S(\bar{\boldsymbol{x}}_i)$ be defined by

$$S(\bar{\boldsymbol{x}}_i) = \{\boldsymbol{x} \in D | Q(\boldsymbol{x}) = \bar{\boldsymbol{x}}_i\},$$

i.e. $S(\bar{\boldsymbol{x}}_i)$ is the set of all points quantized to $\bar{\boldsymbol{x}}_i$, also known as the "Voronoi region" for $\bar{\boldsymbol{x}}_i$. For most distortion measures and quantization schemes of interest, $S(\bar{\boldsymbol{x}}_i)$ can be separated into a "volume" component and a "shape" component, as

$$S(\bar{\boldsymbol{x}}_i) = \left\{ \boldsymbol{x} \in D \mid h_{\bar{\boldsymbol{x}}_i} \left(\frac{\boldsymbol{x} - \bar{\boldsymbol{x}}_i}{V(\bar{\boldsymbol{x}}_i)^{1/n}} \right) < 1 \right\},$$

The work of Anand Subramaniam and Bhaskar Rao is supported in part by QUALCOMM and Nokia under MICRO Grants No. 00-082 and 01-069 respectively.

where $V(\bar{\boldsymbol{x}}_i)$ is the volume of the Voronoi region, given by $V(\bar{\boldsymbol{x}}_i) = \int_{\boldsymbol{x}\in S(\bar{\boldsymbol{x}}_i)} d\boldsymbol{x}$, and thus $\int_{x|h_{\bar{\boldsymbol{x}}_i}(\boldsymbol{x})<1} d\boldsymbol{x} = 1$. In this way, the particular Voronoi region can be expressed completely by the volume $V(\bar{\boldsymbol{x}}_i)$ and the function $h_{\bar{\boldsymbol{x}}_i}(\boldsymbol{x})$ which defines the local shape of the Voronoi region.

In high rate theory, Voronoi regions are often approximated by hyper-ellipsoids, which are defined using the above terminology as

$$h^{ell-\boldsymbol{M}}(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x}}{(\kappa_n)^{-2/n} |\boldsymbol{M}|^{1/n}},$$

where κ_n is the volume of an n dim sphere of radius 1. Spherical regions are defined by $h^{ell-\boldsymbol{I}}(\boldsymbol{x})$.

The goal of a quantizer design is to minimize the expected distortion introduced by quantization, as measured by some error measure E defined by

$$E_d(Q) = \left(\mathcal{E}_{\boldsymbol{x}}\left(d^{r/2}(\boldsymbol{x}, Q(\boldsymbol{x}))\right)\right)^{1/r}, \tag{1}$$

The distortion measure d commonly used for LPC quantization in speech coding is the LSD measure, given by

$$LSD(\boldsymbol{a}, \bar{\boldsymbol{a}}) = \frac{\beta}{\gamma\pi} \int_0^{\gamma\pi} (\ln(|A(\omega)|^2) - \ln(|\bar{A}(\omega)|^2))^2 d\omega,$$

where $\beta = (10/\ln(10))^2$, $A(\omega) = 1 - \sum_{i=1}^{v} a_i e^{j\omega i}$, and $\bar{A}(\omega) = 1 - \sum_{i=1}^{v} \bar{a}_i e^{j\omega i}$. Some work on LPC VQ utilizes "full-band" LSD with $\gamma = 1$ whereas other work focuses on "partial-band" LSD with $\gamma < 1$. In particular, $\gamma = 3/4$, averaging over only the first 3 kHz of the spectrum, is often used. Also, some work focuses on "MS" LSD with $r = 2$, whereas other work focuses on "RMS" LSD with $r = 1$.

In the aforementioned references, it is shown that in the limit as the two vectors approach each other, most distortion functions of interest approach the following limit:

$$d(\boldsymbol{x}, \bar{\boldsymbol{x}}) \to \frac{1}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}})^T \boldsymbol{D}(\bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}})$$

where $\boldsymbol{D}(\bar{\boldsymbol{x}})$ is an n by n dimensional "sensitivity" matrix with j, kth element defined by

$$D_{j,k}(\bar{\boldsymbol{x}}) = \left.\frac{\partial^2 d(\boldsymbol{x}, \bar{\boldsymbol{x}})}{\partial x_j \partial x_k}\right|_{\boldsymbol{x}=\bar{\boldsymbol{x}}}.$$

The diagonal elements of the sensitivity matrix are the scalar sensitivities which represent the degree to which quantization error in a particular scalar parameter increases the overall distortion.

In the references, it is shown that, at high rates, the expression for the expected distortion approaches

$$E_d^r(Q) \to \frac{1}{2^{r/2}} \sum_{\bar{\boldsymbol{x}}_i \in \bar{\boldsymbol{X}}} f(\bar{\boldsymbol{x}}_i) I_r\left[h_{\bar{\boldsymbol{x}}_i}, \boldsymbol{D}(\bar{\boldsymbol{x}}_i)\right] V^{\frac{n+r}{n}}(\bar{\boldsymbol{x}}_i)$$

with I_r is the local expected distortion of a particular, volume-normalized Voronoi region, defined to be

$$I_r\left[h, \boldsymbol{D}\right] = \int_{\boldsymbol{y}:h(\boldsymbol{y})<1} \left(\boldsymbol{y}^T \boldsymbol{D} \boldsymbol{y}\right)^{r/2} d\boldsymbol{y}.$$

I_r is a function of only the shape of the local Voronoi regions and is not a function of the volume of the local regions.

Let $\lambda(\boldsymbol{x})$ be the relative density of codevectors, defined as $\lambda(\boldsymbol{x}) = lim_{B\to\infty} 1/(2^B V(\boldsymbol{x}))$ which integrates to 1. Then as $B \to \infty$,

$$E_d^r(Q) \to \frac{2^{-Br/n}}{2^{r/2}} \int f(\boldsymbol{x}) I_r\left[h_{\boldsymbol{x}}, \boldsymbol{D}(\boldsymbol{x})\right] \lambda^{\frac{-r}{n}}(\boldsymbol{x}) d\boldsymbol{x}.$$

A given quantizer is completely defined by $\lambda(\boldsymbol{x})$, which defines the density of codevector points (and thus also the volume of the local Voronoi regions), and by $h_{\boldsymbol{x}}$, which defines the local shape of the Voronoi regions at point $\boldsymbol{x}$.

For any suitable distortion function of interest, the optimal ellipsoidal Voronoi region shapes can be approximated, the optimal density function given these shapes can be determined, and thus an approximation for the expected optimal distortion can be computed. This allows expressions for the performance of optimal quantizers to be derived, and also allows expressions for the performance of suboptimal quantizers designed using suboptimal distortion measures to be derived.

Following the approach presented in [1], the optimal local Voronoi region shapes can be approximated by

$$h_{\boldsymbol{x}}^{opt}(\boldsymbol{y}) = h_{\boldsymbol{x}}^{ell-\boldsymbol{D}(\boldsymbol{x})}(\boldsymbol{y}),$$

the Voronoi region shapes which result when an unstructured quantizer is trained by minimizing the WMSE can be approximated by

$$h_{\boldsymbol{x}}^{WMSE}(\boldsymbol{y}) = h_{\boldsymbol{x}}^{ell-diag(D_{i,i}(\boldsymbol{x}))}(\boldsymbol{y}),$$

and the Voronoi region shapes which result when an unstructured quantizer is trained by minimizing the MSE can be approximated by

$$h_{\boldsymbol{x}}^{MSE}(\boldsymbol{y}) = h_{\boldsymbol{x}}^{ell-\boldsymbol{I}}(\boldsymbol{y}).$$

These hyper-ellipsoidal approximations for the shapes cannot be fit into a non-overlapping lattice which covers the region of support for the input vectors, and thus the use of these approximate shapes results in lower bound to the truly achievable distortion.

The optimal codevector density function for the quantizer with the optimal Voronoi region shapes can then be approximated [1] as

$$\lambda_{opt}(\boldsymbol{x}) = \frac{\left(I_r[h_{\boldsymbol{x}}^{opt}, \boldsymbol{D}(\boldsymbol{x})] f(\boldsymbol{x})\right)^{n/(n+r)}}{\int \left(I_r[h_{\boldsymbol{y}}^{opt}, \boldsymbol{D}(\boldsymbol{y})] f(\boldsymbol{y})\right)^{n/(n+r)} d\boldsymbol{y}},$$

and similarly the codevector density functions which result when an unstructured quantizer is trained by minimizing the WMSE and MSE, $\lambda_{WMSE}(\boldsymbol{x})$ and $\lambda_{MSE}(\boldsymbol{x})$ can be defined using the above expression with h^{WMSE} and h^{MSE} respectively replacing h^{opt}.

With the optimal approximation described above, it can be shown that

$$I_r[h_{\boldsymbol{x}}^{opt}, \boldsymbol{D}(\boldsymbol{x})] = \frac{n|\boldsymbol{D}(\boldsymbol{x})|^{r/2n}}{(n+r)\kappa_n^{(r/n)}}.$$

For $r = 2$, a closed form expression exists for I for an arbitrary ellipsoidal shape [1],

$$I_2[h_{\boldsymbol{x}}^{ell-\boldsymbol{M}(\boldsymbol{x})}, \boldsymbol{D}(\boldsymbol{x})] = \frac{|\boldsymbol{M}(\boldsymbol{x})|^{1/n} tr\left(\boldsymbol{M}^{-1}(\boldsymbol{x})\boldsymbol{D}(\boldsymbol{x})\right)}{(n+2)\kappa_n^{(2/n)}},$$

and thus closed form expressions for the optimal, WMSE, and MSE quantizers exist. Unfortunately, we know of no closed form expression for an arbitrary shape for $r \neq 2$, although the integral can be numerically evaluated or approximated.

Given a database of vectors, the approximate expressions for the expected distortion can be approximated by a sum over the vectors in the database as

$$E_d^r(Q) \approx \frac{2^{-Br/n}}{2^{r/2}} \sum_{\boldsymbol{x}} I_r\left[h\boldsymbol{x}, \boldsymbol{D}(\boldsymbol{x})\right] \lambda^{-r/n}(\boldsymbol{x}).$$

As shown in the above expressions, the codevector density, $\lambda(\boldsymbol{x})$, is typically a normalized function of $f(\boldsymbol{x})$ to some power $\neq 1$, so evaluating this sum requires a good model of the density of the input. In many of the references and in this work, the previously defined Gaussian mixture model is used as an estimate of $f(\boldsymbol{x})$.

Similar derivations which utilize recursion in the quantization procedure has been described in [4]. In this approach, the source is taken as a random vector sequence $\boldsymbol{x}_i$ and the quantizer is allowed to change at each discrete time i as a function of previous input vectors $\boldsymbol{x}_{i-1}$ to $\boldsymbol{x}_{i-N}$. Space does not permit listing the equivalent expressions, but the derivation in [4] combined with the formulation above is straightforward.

3. EVALUATION OF THEORETICAL ESTIMATES

This section describes methods used to evaluate the expressions for the high rate performance and provides the resulting values for a comprehensive set of conditions.

3.1. Procedures

The numerical estimates were computed using a database of 150,000 spectral vectors. Tenth order LPC coefficients were computed from a speech database every 20 ms using standard Levinson-Durbin recursions with bandwidth expansion of 10 Hz.

Estimation of the density of the spectral vectors was performed in the LSP domain with N=1 (1st order recursion, modeling 20 dimensional vectors $[\boldsymbol{x}_{i-1}\boldsymbol{x}_i]$) using a 32 mixture model allowing full optimization of the mean vector and covariance matrix per mixture. The optimization was performed over 600,000 LSP vectors which were distinct from the 150,000 used in evaluation of the distortion expressions. A non-recursive (N=0) model was computed directly from the N=1 model parameters. The bounded support region of the LSP vectors was not taken into account during EM optimization as in [6], but the final density estimate was divided by the probability that vectors generated with the resulting estimated density function was within the region of bounded support.

For other, non-LSP parameter types, the density function was transformed from the Gaussian mixture model of the LSPs to the other parameter set via $f_{Param}(\boldsymbol{p}) = \frac{1}{|J\boldsymbol{\omega}|} f\boldsymbol{\omega}(\boldsymbol{\omega})$, where J is the Jacobian matrix of the transformation from the LSP domain to the other parameter set. This approach ensures that the same density model is used for all parameter sets and that no bias between parameter sets exists due to differences in modeling.

As noted in [4], practical normalization of the density functions requires averaging over randomly generated vectors rather than over the database of spectral vectors. For non-recursive results, this normalization factor need only be computed once: for these cases 10,000 random vectors were used for computing the normalization value. For recursive results, each input vector has a different set of previous $\boldsymbol{x}_{i-1}...\boldsymbol{x}_{i-N}$ leading to a different optimal codevector density and thus a different codevector density normalization factor. Due to the high complexity involved, only 100 random vectors were used for each input vector in the normalization procedure. It was found that increasing this number above 100 did not affect the results presented by more than approximately 0.1 bit.

3.2. Results

The distortion integral is a function of the number of bits used. The tables presented below list the number of bits required to achieve an average LSD of 1 dB.

Table 1 shows the high rate estimate of the bits necessary for the optimal (i.e., using $h_{\boldsymbol{x}}^{opt}$ and $\lambda_{opt}(\boldsymbol{x})$ in the performance expressions) non-recursive RMS quantizer ($r = 1$) and MS quantizer ($r = 2$) to achieve 1 dB LSD. The distortion for the optimal quantizer is identical for all parameter types. For *optimal* quantizers, partial-band LSD requires close to 2 bits less than that of full-band LSD.

Table 2 shows the high rate estimate of the bits necessary for the non-predictive MS ($r = 2$) LSP and CEPS quantizers trained and tested using the true LSD (i.e., optimal with $h_{\boldsymbol{x}}^{opt}$ and $\lambda_{opt}(\boldsymbol{x})$), the WMSE (i.e., with $h_{\boldsymbol{x}}^{WMSE}$ and $\lambda_{WMSE}(\boldsymbol{x})$), and MSE (i.e., with $h_{\boldsymbol{x}}^{MSE}$ and $\lambda_{MSE}(\boldsymbol{x})$) to achieve 1 dB full-band and partial-band LSD. While the optimal quantizer requires fewer bits for partial-band LSD, the MSE quantizer does not. This is because the partial-band weighting causes the LSP sensitivity matrix to not be diagonal, and causes the higher LSPs to become less sensitive, resulting in the diagonal elements becoming more separated and thus causing the identity matrix approximation used in the MSE case to become a poorer approximation. Similar effects are seen for the CEPS parameters.

Table 3 shows the high rate estimate of the bits necessary for the non-predictive MS ($r = 2$) quantizers trained and tested by minimizing the true LSD, the WMSE, and the MSE to achieve 1 dB full-band LSD. These results are for the LPC coefficients, the reflection or PARCOR coefficients (K), the Log Area Ratio (LAR) and ArcSine (ASIN) transforms of the reflection coefficients, the Line Spectral Pair (LSP) frequencies, and the Cepstral coefficients (CEPS) [8]. The LSP pairs have a diagonal $\boldsymbol{D}$ matrix for full band LSD, and so the LSP performance is optimal also for the WMSE quantizer [1]. The LSP performance is also good for full-band MSE. The LAR and ASIN coefficients are scalar transformations of the K parameters. As a result, the WMSE performance for all 3 parameter types is identical and only MSE performance is better with LARs and ASINs vs. Ks.

Table 4 shows the high rate estimate of the bits necessary for the first-order predictive MS ($r = 2$) quantizers trained and tested by minimizing the true LSD, the WMSE, and the MSE to achieve 1 dB full-band LSD for the same parameters shown in the previous table. 1st order recursion results in a savings of 3.5 bits.

Tables 5 and 6 show the high rate estimate of the bits necessary for the non-predictive MS ($r = 2$) of a set of LSP and CEPS quantizers with the optimal, WMSE, and MSE Voronoi region shapes, crossed with the optimal, WMSE, and MSE density functions, to achieve a 1 dB full-band LSD. The columns show the results using $h_{\boldsymbol{x}}^{opt}$, $h_{\boldsymbol{x}}^{WMSE}$, and $h_{\boldsymbol{x}}^{MSE}$, while the rows show the results using $\lambda_{opt}(\boldsymbol{x})$, $\lambda_{WMSE}(\boldsymbol{x})$, $\lambda_{MSE}(\boldsymbol{x})$, and $\lambda(\boldsymbol{x}) = f(\boldsymbol{x})$. Similar results were obtained for 1st order recursive quantizers, but space limitations do not allow the results to be included here. These estimates do not appear to be particularly sensitive to the density function. In particular, utilizing a suboptimal density function with the optimal Voronoi region shape appears to also achieve good performance. However, these estimates are highly sensitive to the Voronoi region shape.

The approach here can be used to evaluate the high rate performance of an arbitrary deterministic VQ scheme where the statistics of the Voronoi region shapes and the codevector density can be computed. For example, future work may include evaluating the performance of the practical schemes described in [4, 7].

The "optimal" results here assume that the source density modeled by the Gaussian mixture model is the true density of the source.

This is an overly optimistic assumption, and the "optimal" results presented here are an upper bound to the estimate which would be computed if the true LSP source density was known. Improvements in modeling (e.g., by using a greater number of mixtures, incorporating the bounded support during the EM optimization as in [6], etc.) may result in further refinements and improved estimates.

4. CONCLUSION

High rate VQ theory has been reformulated to separate the effects of codevector density and local Voronoi region shape on overall VQ performance. Estimates of the theoretical performance for a large class of spectral quantizers have been presented.

Future work may include improving the modeling of the LSP source density, analysis of recursive quantizers of order greater than one, and analysis of quantizers designed with arbitrary but specific and/or random Voronoi regions shapes and codevector densities (e.g., the quantizers discussed in [4, 7]).

5. REFERENCES

[1] W. R. Gardner and B. D. Rao, "Theoretical Analysis of the High Rate Vector Quantization of LPC Parameters," *IEEE Trans. on Speech and Audio.*, 1995.

[2] J. Li, N. Chadda, and R. M. Gray, "Asymptotic Performance of Vector Quantizers with a Perceptual Distortion Measure," *IEEE Trans. on Information Theory*, May 1999.

[3] T. Linder, R. Zamir, and K. Zeger, "High Resolution Source Coding for Non-Difference Distortion Measures: Multi-Dimensional Companding," *IEEE Trans. on Information Theory*, March 1999.

[4] J. Samuelsson and P. Hedelin, "Recursive Coding of Spectrum Parameters," *IEEE Trans. on Speech and Audio.*, July 2001.

[5] P. Hedelin, J. Skoglund, and J. Samuelsson, "Performance Bounds for LPC Spectrum Quantization," in *Proceedings of ICASSP*, 1999.

[6] P. Hedelin and J. Skoglund, "Vector Quantization Based on Gaussian Mixture Models," *IEEE Trans. on Speech and Audio.*, July 2000.

[7] A. D. Subramaniam and B.D. Rao, "Speech LSF Quantization with Rate Independent Complexity, Bit Scalability, and Learning," in *Proceedings of ICASSP*, 2001.

[8] L. Rabiner and R. Schafer, *Digital Processing of Speech Signals*, Prentice-Hall, 1978.

	$\gamma = 1$	$\gamma = 0.75$
$r = 1$	21.96	20.08
$r = 2$	22.45	20.64

Table 1. Bits Required for E = 1 dB LSD for Optimal Quantizers vs. r, full/partial band

Param	Bandwidth	Optimal	WMSE	MSE
LSP	$\gamma = 1$	22.45	22.45	23.46
LSP	$\gamma = 0.75$	20.64	20.80	23.96
CEPS	$\gamma = 1$	22.45	26.43	28.28
CEPS	$\gamma = 0.75$	20.64	27.23	29.55

Table 2. Bits Required for E = 1 dB MS ($r = 2$) LSD for Optimal LSP/CEPS Quantizers vs. full/partial band

Param	Opt	WMSE	MSE
LPC	22.45	32.41	32.62
K	22.45	27.54	29.88
ASIN	22.45	27.54	29.09
LAR	22.45	27.54	29.12
LSP	22.45	22.45	23.46
CEPS	22.45	26.43	28.28

Table 3. Bits Required for E = 1 dB MS ($r = 2$) full-band ($\gamma = 1$) LSD vs. parameter type, error measure used in training/testing : no recursion

Param	Optimal	WMSE	MSE
LPC	17.02	26.54	26.65
K	17.02	21.82	24.17
ASIN	17.02	21.82	23.41
LAR	17.02	21.82	23.43
LSP	17.02	17.02	17.99
CEPS	17.02	20.38	22.39

Table 4. Bits Required for E = 1 dB MS ($r = 2$) full-band ($\gamma = 1$) LSD vs. parameter type, error measure used in training/testing : 1st order recursion

Dens - Shape	Optimal	WMSE	MSE
Optimal	22.45	22.45	23.49
WMSE	22.45	22.45	23.49
MSE	22.47	22.47	23.46
$f(\boldsymbol{x})$	22.67	22.67	23.73

Table 5. Bits Required for LSP Quantizers for E = 1 dB MS ($r = 2$) full-band ($\gamma = 1$) LSD vs. combination of density and Voronoi region shape

Dens - Shape	Optimal	WMSE	MSE
Optimal	22.45	26.06	28.21
WMSE	23.07	26.43	28.45
MSE	22.46	26.09	28.28
$f(\boldsymbol{x})$	22.67	26.35	28.53

Table 6. Bits Required for CEPS Quantizers for E = 1 dB MS ($r = 2$) full-band ($\gamma = 1$) LSD vs. combination of density and Voronoi region shape