# DOA ESTIMATION PERFORMANCE BREAKDOWN: A NEW APPROACH TO PREDICTION AND CURE

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### ABSTRACT

The well-known performance breakdown of subspace-based parameter estimation methods is usually attributed to a specific property of the technique, namely "subspace swap". In this paper, we derive the lower bound for the maximum likelihood ratio (LR), and use it as a simple data-based indicator to determine whether or not any set of estimates could be treated as a maximum likelihood (ML) set. We demonstrate that in those cases where the performance breakdown is subspace specific, this LR analysis provides reliable identification of whether or not "subspace swap" has actually occurred. We also demonstrate that by proper LR maximisation, we can extend the range of signal-to-noise ratio (SNR) values and/or number of data samples wherein accurate parameter estimates are produced. Yet, when the SNR and/or sample size falls below a certain limit for a given scenario, we show that ML estimation suffers from a discontinuity in the parameter estimates, a phenomenon that cannot be eliminated within the ML paradigm.

## 1 INTRODUCTION

All subspace-based parameter estimation techniques are known to suffer a rapid degradation in performance as the SNR and/or the number of snapshots $N$ drop below certain threshold values [1, 2, 3, 4]. The explanation of this phenomenon has involved a discontinuity in parameter estimates that are specific to the technique in question. In particular, the sole apparent discontinuity (that is typical for all subspace-based methods) is induced by the interchange of vectors between the estimated signal and noise subspaces ("subspace swap") [4]. Investigations into these subspace threshold conditions have been conducted in [1, 2, 3], with some recent attempts to determine from the data whether or not a subspace swap has actually occurred [4]. The latter paper also proposes a method of "curing" performance breakdown by comparing the deterministic ("concentrated") likelihood over a set of different partitionings of the signal- and noise-subspace eigenvectors.

However, the important question of whether the true ML estimation (that does not have any discontinuous assignment of eigenvectors in the signal and noise subspaces) can still suffer performance breakdown has not been directly addressed so far. Obviously, a positive answer to this question defines the ultimate limit (in SNR and/or $N$), beyond which accurate estimation is not possible.

In this paper, we demonstrate that the LR for the true parameters (*ie.* the exact covariance matrix) does not depend on the particular scenario, and therefore could be used as a (statistical) lower bound for ML estimation. We suggest that a comparison of the LR generated by any given set of parameter estimates with this lower bound is used to identify non-ML estimates, and in particular subspace-specific "outliers". We demonstrate that for certain threshold conditions, LR maximisation can result in parameter estimation with outliers that nevertheless still generates LRs that exceed those produced by the true (exact) parameters. Obviously, any "discontinuous" set of estimates that is better than the true set of parameters (in terms of LR) could not be identified as being discontinuous, and could not be improved upon with the entire ML paradigm, and therefore the above-mentioned threshold conditions for LR maximisation are treated as the ultimate ones.

Simulation results for direction-of-arrival (DOA) estimation are introduced to support these ideas.

## 2 PROBLEM FORMULATION

Consider an arbitrary linear antenna array with $M$ sensors located at positions

$$\boldsymbol{d} \equiv [d_1 \equiv 0,\, d_2,\, \ldots,\, d_M] \tag{1}$$

measured in terms of half-wavelength units. We assume that Gaussian processes are observed as a combination of $m$ uncorrelated plane waves with DOAs $\boldsymbol{\theta} \equiv [\theta_1, \ldots, \theta_m]^T$, powers $\boldsymbol{p} \equiv \mathrm{diag}\,[p_1, \ldots, p_m]^T$ and Gaussian white noise of power $p_0$:

$$\boldsymbol{y}(t) = S(\boldsymbol{\theta})\, \boldsymbol{x}(t) + \boldsymbol{\eta}(t) \qquad \text{for} \quad t = 1, \ldots, N \tag{2}$$

where $\boldsymbol{y}(t) \in \mathcal{C}^{M\times 1}$ is the vector of observed sensor outputs (the "snapshot"), $\boldsymbol{x}(t) \in \mathcal{C}^{m\times 1}$ is the vector of

Gaussian signal amplitudes

$$\mathcal{E}\{\boldsymbol{x}(t_1)\,\boldsymbol{x}^H(t_2)\} = \begin{cases} P \equiv \operatorname{diag}[\boldsymbol{p}] & \text{for } t_1 = t_2 \\ 0 & \text{for } t_1 \neq t_2\,, \end{cases} \quad (3)$$

and $\boldsymbol{\eta}(t) \in \mathcal{C}^{M\times 1}$ is white Gaussian noise. The array-signal manifold matrix is $S(\boldsymbol{\theta}) \equiv [\boldsymbol{s}(\theta_1), \ldots, \boldsymbol{s}(\theta_m)] \in \mathcal{C}^{M\times m}$, where each

$$\boldsymbol{s}(\theta_j) = \left[1,\, \exp\left(i\pi d_2 \sin\theta_j\right),\, \ldots,\, \exp\left(i\pi d_M \sin\theta_j\right)\right]^T \quad (4)$$

is a so-called steering vector. The set of independent snapshots $\boldsymbol{y}(t) \in \mathcal{C}^{M\times 1}$ originates from a complex Gaussian distribution $\mathcal{CN}(M, 0, R)$, where

$$R \,=\, S(\boldsymbol{\theta})\, P\, S^H(\boldsymbol{\theta}) + p_0\, I_M\,. \quad (5)$$

Given $N$ independent snapshots, the sufficient statistic for DOA estimation is the DDC (sample) matrix

$$\hat{R} \,=\, \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{y}(t)\,\boldsymbol{y}^H(t)\,. \quad (6)$$

## 3 LOWER BOUND FOR MAXIMUM LR

Let $\hat{\boldsymbol{\theta}}_\mu$, $\hat{\boldsymbol{p}}_\mu$ and $\hat{p}_0$ respectively be the DOA, power and noise power estimates that together are the set of parameters uniquely defining the model for the observed signal data $\boldsymbol{y}(t)$ involving $\mu$ plane-wave sources. Then the sphericity test for the hypothesis

$$\begin{array}{lll} H_0: & \mathcal{E}\left\{R_\mu{}^{-\frac{1}{2}}\,\hat{R}\,R_\mu{}^{-\frac{1}{2}}\right\} = c_0\, I_M & \text{against} \\ H_1: & \mathcal{E}\left\{R_\mu{}^{-\frac{1}{2}}\,\hat{R}\,R_\mu{}^{-\frac{1}{2}}\right\} \neq c_0\, I_M\,, & c_0 > 0 \end{array} \quad (7)$$

could be used to accept or reject this model. For Gaussian mixtures, the LR for this test is

$$\gamma(R_\mu) \,=\, \left(\frac{\det(R_\mu{}^{-1}\hat{R})}{\left[\frac{1}{M}\operatorname{tr}(R_\mu{}^{-1}\hat{R})\right]^M}\right)^N \,\equiv\, \gamma_0^N(R_\mu) \quad (8)$$

$0 < \gamma(R_\mu) \leq 1$, with the set of estimated parameters $\{\hat{\boldsymbol{\theta}}_\mu, \hat{\boldsymbol{p}}_\mu, \hat{p}_0\}_{ML}$ being those that yield the global maximum of the function $\gamma(R_\mu)$. Then information-theoretic or Bayesian criteria may be used for model selection.

Note that the global maximum of the LR function (8) is identically equal to that of the likelihood function

$$\mathcal{L}(R_\mu) = \min_\sigma N\left\{\log\det(\sigma R_\mu) + \operatorname{tr}[(\sigma R_\mu)^{-1}\hat{R}]\right\}\,, \quad (9)$$

since this immediately gives us $\sigma_{ML} = \operatorname{tr}(R_\mu{}^{-1}\hat{R})/M$ and therefore for $R_\mu > 0$

$$\mathcal{L}(R_\mu) \,=\, \log\left\{\left[\frac{\operatorname{tr}(R_\mu{}^{-1}\hat{R})}{M}\right]^M \frac{M}{\det R_\mu{}^{-1}}\right\}^N\,. \quad (10)$$

For a nonsingular sample matrix $\hat{R}$ (guaranteed for $N \geq M$), the minimum argument of $\mathcal{L}(R_\mu)$ coincides with the maximum argument of $\gamma(R_\mu)$; thus there is no difference between maximum LR and ML estimates for $R_\mu$.

Our lower bound originates from the straight-forward observation that, for any $\mu \geq m$, the exact covariance matrix $R$ of the given model obviously belongs to the admissible set, thus for any given $\hat{R}$ with $N \geq M$ and $\mu \geq m$, a proper LR maximisation must yield a solution $R_\mu$ such that $\gamma(R_\mu) \geq \gamma(R)$. This observation would be of academic interest only, since the exact covariance matrix $R$ is unknown in practical applications, except that $\gamma(R)$ *does not depend on* $R$. Indeed, according to (8),

$$\gamma_0(R) = M^M \det\hat{G}\left[\operatorname{tr}\hat{G}\right]^{-M} \quad (11)$$

where $\hat{G}(R) \equiv R^{-\frac{1}{2}}\hat{R}\,R^{-\frac{1}{2}}$ is a random matrix with complex Wishart distribution $\mathcal{CW}(N, M, I_M)$, completely defined by the sample support $N$ and the number of sensors $M$ ($N \gg M$), since $\mathcal{E}\{\hat{G}\} = I_M$.

## 4 SIMULATION RESULTS

We address the question of how well LR bound analysis can predict the performance breakdown of subspace techniques, and to investigate a possible curable domain in threshold conditions between the edge of subspace techniques and the boundary of ML techniques. We reiterate that beyond the performance breakdown boundary of ML techniques, where extremely erroneous matrices have a higher LR than the exact covariance matrix, no possible remedy is expected.

Consider the scenario comprising a five-sensor uniform linear array $\boldsymbol{d}_5$ with three sources $\boldsymbol{w}_3$. We conduct our investigation of performance breakdown conditions in the case of the moderate sample volume $N = 100$. At the comparatively high SNR of 20 dB per source, we systematically vary the third of the three DOAs:

$$\boldsymbol{w}_3 = [-0.40,\, 0.00,\, w_3] \quad (12)$$

with $w_3 = \{0.03, 0.04, 0.05, 0.06, 0.08, 0.10\}$ so that the range of source separations is adequately covered, and where $w \equiv \sin\theta$. For each scenario, we calculate (a) the LR of the MUSIC-derived covariance matrix, computing the source power estimates $\hat{\boldsymbol{p}}$ and the white noise power estimate $\hat{p}_0$ from the traditional DOA estimates $\hat{\boldsymbol{\theta}}$; (b) $\gamma(R)$, where $R$ is the exact covariance matrix; and (c) $\gamma(R_{ML})$, where $R_{ML}$ is the (local) ML estimate of the $M$-variate p.d. Toeplitz covariance matrix with the $(M-m)$ smallest eigenvalues being equal (the computational details for LR maximisation appear in [5, 6]). Additionally, in each of the 1000 Monte-Carlo trials, these three estimators are each determined to be (i) a normal DOA estimate or an outlier ("correct identification"); and (ii) ML-optimisation successful or not. Since our LR maximisation algorithm does not guarantee reaching the global extremum and depends on successful initialisation, success for ML purposes is defined according to the condition $\gamma(R_{ML}) \geq \gamma(R)$. If this condition is not met, the estimate cannot be a ML estimate. While these trials should be excluded from the experimental statistics, in practical situations the value $\gamma(R)$ is of course unknown, so we report on the overall success rate of ML optimisation. For unsuccessful ML optimisation, we also identify (iii) those trials that incorrectly identify the DOAs based on $R_{ML}$. Finally, we also determine (iv) the error of the worst DOA estimate, in order to compare this with the Cramér–Rao bound.

Figs. 1–3 show these LR and error distributions for a representative subset of the above source separations.

Fig. 1 begins with the benign widely separated scenario $w_3 = 0.10$. We see that the LRs for $R_{MUSIC}$, $R$ and $R_{ML}$ are statistically similar, though it is clear that the LR-maximised solutions $R_{ML}$ are "better" than the true covariance matrix (in terms of LR); this "improvement" does not involve DOA estimation accuracy improvement. The two MUSIC estimates here with an extremely low LR are easily classified as outliers. Note that the distribution of $\gamma(R)$ is scenario free.

Fig. 2 shows results for the separation $w_3 = 0.06$, chosen to demonstrate an example where the probability of correct identification by MUSIC is about one half, in fact, the MUSIC performance breakdown rate here is 0.535. As expected, the LR distribution for $R_{MUSIC}$ has two widely separated peaks: the peak with extremely low LRs is entirely due to outliers, while the peak with optimally high LRs is due to the 465 correctly identified trials. For ML estimation, $R_{ML}$ still has mainly high LR values, above $\gamma(R)$. Specifically, only in 45 trials was $\gamma(R_{ML})$ smaller than $\gamma(R)$. Interestingly, as seen in the bottom-left subfigure, only six of these 45 trials gave very low LRs; the remaining 20 outliers have relatively high LR and therefore could not be determined as inappropriate estimates in practice. In fact, only 12 of the 26 outliers were among the unsuccessfully optimised solutions, while in the 14 other cases these outliers had LRs exceeding those of the true covariance matrix. This example demonstrates the ability of our technique to reliably determine subspace-specific outliers and to efficiently rectify incorrect estimates by accurate ML optimisation. This is possible here because there is a "gap" between the threshold conditions for MUSIC and those for ML estimation. DOA estimation accuracy is significantly improved by LR maximisation compared with MUSIC, since the outliers have been rectified.

Our final example with $w_3 = 0.03$ (Fig. 3) was chosen to represent threshold conditions for ML estimation. Here, MUSIC fails completely, with a sample probability of correct identification of only 0.019. The top-right subfigure indicates that most of the incorrectly identified trials have a low LR, and so could be reliably determined as improper estimates. Our $R_{ML}$ solution has a probability of correct identification of 0.648. We emphasise that the success rate of ML optimisation is significantly higher at 0.893, with only 26% of the 352 incorrectly identified $R_{ML}$ trials also having a LR less than that of $R$ (*ie.* unsuccessfully optimised). The vast majority of outliers still had LRs greater than that of $R$. The bottom-left subfigure leaves no doubt that these outliers, that are "better than" the exact DOAs, could not be determined nor rectified ("neither predicted nor cured"). Here we have approached the continuity threshold for ML parameter estimation. Nevertheless even here, most MUSIC outliers could be determined, and more than half of them rectified.

## 5 SUMMARY AND CONCLUSIONS

We have demonstrated that performance breakdown of subspace-based techniques such as MUSIC could be reliably determined by LR analysis. Subspace-specific "outliers" generate LR values that are significantly smaller than the introduced LR lower bound. Proper LR optimisation is proven to significantly improve the breakdown threshold, extending it to the ultimate point where breakdown is intrinsic to the true ML solution. Indeed, incorrect solutions (outliers) that still have LRs exceeding that of the true covariance matrix could not be diagnosed nor cured within the ML paradigm.

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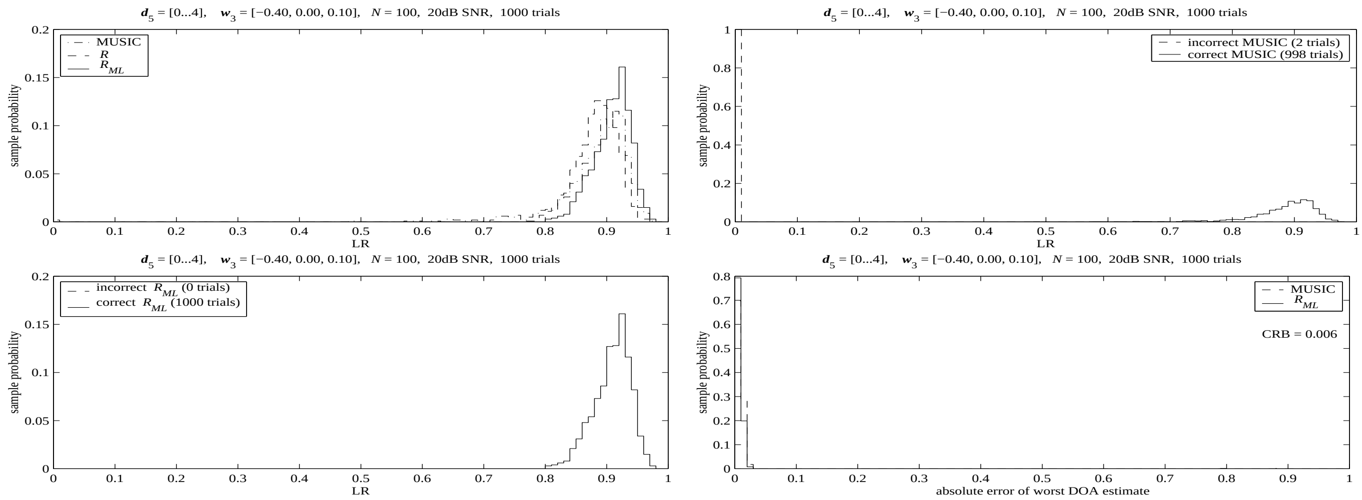


Figure 1: LR analysis results for $w_3 = 0.10$.

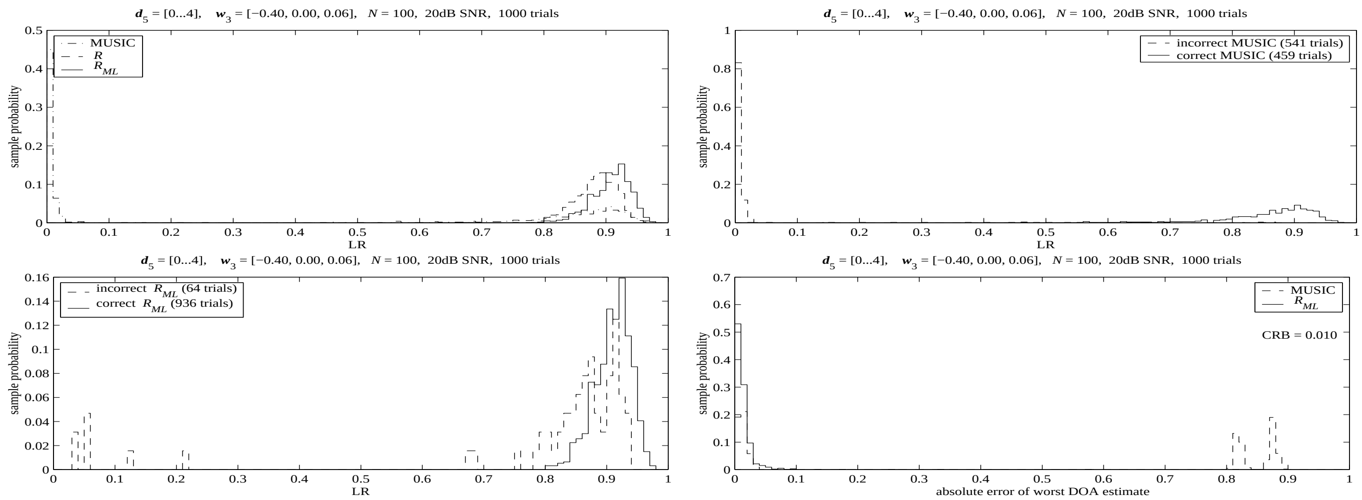


Figure 2: LR analysis results for $w_3 = 0.06$.

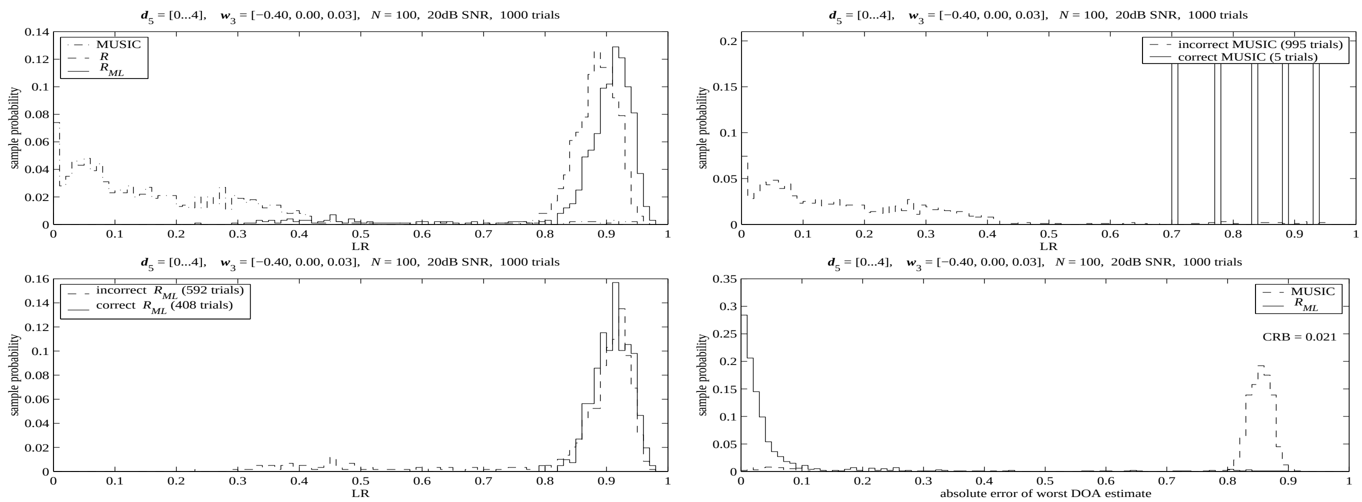


Figure 3: LR analysis results for $w_3 = 0.03$.