Estimation of the symbol period: the frequency offset case

Sebastien Houcke[1], *Antoine Chevreuil*[1], *Philippe Ciblat*[2]
[1]Université de Marne la Vallée, [2] ENST, COM ELEC
e-mail: houcke@univ-mlv.fr, chevreui@univ-mlv.fr, ciblat@com.enst.fr

ABSTRACT

We address the blind equalization of a telecommunication signal, the characteristics of which are totally unknown (channel, symbol period, alphabet of the symbols, frequency offset). In [1] was proposed a methodology originating from deconvolution ideas, for jointly estimating the symbol period and the emitted symbols. In this contribution, we show that the presence of a frequency offset does not alter this approach. Simulation results illustrate the robustness of the T_s estimate with respect to the frequency offset.

1 Introduction and notations

In the field of digital communication, we consider the following base-band model

$$y(t) = (\sum_k s_k h(t - kT_s))e^{2i\pi\delta f_0 t}$$

where the symbol sequence $(s_n)_{n\in\mathbb{Z}}$ is an independent, identically distributed sequence of symbols, T_s is the duration of a symbol, δf_0 a frequency offset and h the impulse response of a band-limited filter resulting from the multi-path channel and a shaping filter. We address the following equalization problem: recover $(s_n)_{n\in\mathbb{Z}}$, or a shifted/scaled version, only from the observation y. This kind of problem is called blind since the channel h, the symbol period T_s, the frequency offset δf_0 are unknown. In particular, no training sequence is available. Such a problematic arises for instance in passive listening. Undoubtedly, the major difficulty lies in the estimation of T_s, since the standard equalization methods rely on a sampled version of y at rate k/T_s, k integer. In this respect, one notices that y is cyclostationary, its cyclic frequencies being multiples of $1/T_s$: this remark is the keystone of many estimates of T_s, which basically identify a cyclic frequency from a sampled version of the observation. However, these estimates may suffer from severe numerical problems as soon as the transmission bandwidth approaches the limit $1/T_s$ (i.e. when the excess bandwidth goes to zero), especially the second-order estimates [2][3]. As far as the higher-order estimates are concerned [4], they often require large amounts of data in order to be accurate, hence limiting their applicability, especially in the context of rapidly varying environments (e.g. mobile systems).

A novel methodology was introduced in [1], when $\delta f_0 = 0$: in particular, an estimate of the symbol period is provided, its main characteristics being that 1) it is robust to a lack of excess bandwidth (actually the less the excess bandwidth factor, the better the performance), and 2) it requires small amounts of data to provide an accurate estimate. Basically, this approach relies on deconvolution ideas. It is there proposed to extend the concept of contrast functions, such as the Kurtosis introduced in [5] in a cyclostationary framework. We recall informally the methodology. Consider a given sampling period T and the associated cyclo-stationary discrete-time signal $(y(nT) = y_n)_{n\in\mathbb{Z}}$. As is usual in the deconvolution framework, it is proposed to pass this time-series through a digital filter G: denote $(z_n)_{n\in\mathbb{Z}}$ the (still cyclo-stationary) output time-series. We inquire about the parameters G and T for which $(z_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence. More practically, a class of functions "contrast functions" (hence depending on the parameters G, T) involving certain statistics of $(z)_{n\in\mathbb{Z}}$ is introduced: a function of this class is lower-bounded, the lower bound being reached if and only if $T = T_s$ and the filter G is a scaled/delayed inverse of the channel, i.e. $z_n = s_n$ up to a scale/delay. Clearly, this argument allows to identify T_s via an optimization procedure. Among the contrast functions introduced in [1], we focus, in this paper, on the one derived from the Kurtosis contrast function, namely

$$\frac{< E|z_{n,T}|^4 >}{(< E|z_{n,T}|^2 >)^2} \quad (1)$$

where $< . >$ denotes the temporal average. In this paper, we prove that the above cost function remains a contrast in the case of a non-null frequency offset δf_0. In particular, the optimization of this function in the parameters (G, T) allows to identify the unknown symbol duration T_s.

The paper is organized as follows. Recalling that the contrast property of (1) *in the particular case* $\delta f_0 = 0$

mainly stems from the contrast property of (1) when $T = T_s$, we work out in Section 2 the deconvolution problem when $T = T_s$ and $\delta f_0 \neq 0$. The extension for any T is addressed in Section 3. In Section 4, an algorithm is proposed and simulation results illustrate the good performance of the estimate of T_s, and the robustness with respect to the frequency offset.

2 Particular case of a known T_s

The observation y may be sampled at the rate $1/T_s$, hence providing the time series $(y_n)_{n\in\mathbb{Z}}$ given by

$$y_n = \left(\sum_k h_k s_{n-k}\right) e^{i2\pi\delta f_0 n T_s} \quad (2)$$

$$= \sum_k \tilde{h}_k \tilde{s}_{n-k} \quad (3)$$

where $h_k = h(kT_s)$, $\tilde{h}_k = h_k e^{i2\pi\delta f_0 k T_s}$ and $\tilde{s}_k = s_k e^{i2\pi\delta f_0 k T_s}$. In particular, (3) shows that $(y_n)_{n\in\mathbb{Z}}$ is the output of a filter driven by the independent but in general[1] *non-identically distributed* sequence $(\tilde{s}_n)_{n\in\mathbb{Z}}$. It may therefore be asked if the deconvolution approach relying on the minimization of cost functions may still apply in this more general framework. In this respect, we consider a filter $G = (g_k)$ the time-series $(y_n)_{n\in\mathbb{Z}}$ is passed through, and denote $(z_n)_{n\in\mathbb{Z}}$ the output of the filter; we have

$$z_n = \left(\sum_k f_k s_{n-k}\right) e^{i2\pi\delta f_0 n T_s},$$

where $f_k = \sum_l g_l e^{-i2\pi\delta f_0 T_s l} h_{k-l}$. In the sequel, F denotes $(f_k)_{k\in\mathbb{Z}}$. Let us focus our attention onto the cost function introduced in [5], namely, we consider

$$C_1(G) = \frac{E|z_n|^4 - |Ez_n^2|^2}{(E|z_n|^2)^2}.$$

Notice that C_1 is properly defined in the sense that C_1 does not depend on the time lag n. It is straight-forward to develop $C_1(G)$ using the multi-linearity of the cumulants; it yields

$$C_1(G) = \kappa_s \frac{\sum_k |f_k|^4}{(\sum_k |f_k|^2)^2} + 2,$$

where we let κ_s be the Kurtosis of $(s_n)_{n\in\mathbb{Z}}$ assumed in the sequel sub-Gaussian, i.e. $\kappa_s \leq 0$ (this assumption holds in most telecommunication cases). It is then immediately noticed that $C_1(G) \geq \kappa_s + 2$, with equality if and only if $f_k = 0$ for every k except one. In the following, such trivial filters F will be called *type I* filters. In other word, $C_1(G)$ reaches its minimum if and only if $(z_n)_{n\in\mathbb{Z}}$ coincides with $(s_n e^{i2\pi\delta f_0 n T_s})_{n\in\mathbb{Z}}$ up to a scale and a delay, hence achieving more or less the blind equalization (the detection/compensation of the frequency offset from this z_n is out of the scope of the paper).

We emphasize that this approach might be practically inappropriate in general. In this respect, we first notice that $Ez_n^2 = 0$ when the sequence $(s_n)_{n\in\mathbb{Z}}$ is **complex circular**; in this case, $C_1(G)$ reduces to

$$C_2(G) = \frac{E|z_n|^4}{(E|z_n|^2)^2}, \quad (4)$$

and it is simple to implement the theoretical contrast-based approach. Indeed, when y is observed during T_0, $C_2(G)$ may be consistently estimated from the available data, using empirical estimates for $E|z_n|^4$ and $E|z_n|^2$.

On the contrary, the **non-circular case** is more problematic. Indeed, the term Ez_n^2 depends on the time lag n, hence $|Ez_n^2|^2$ cannot be estimated consistently except if the complex exponential $e^{i2\pi\delta f_0 n T_s}$ is compensated: as the parameter δf_0 is assumed unknown, we have to give up such a possibility. Actually, one may estimate consistently the Fourier coefficient $\lim_{n\to\infty} \frac{1}{N} Ez_n^2$. Unfortunately, this coefficient is 0 except when $\delta f_0 T_s$ is null (modulo 2π). This remark shows that, in the non-circular case, the proper cost function to consider is C_2 given in Equation (4) rather than C_1. As for C_1, C_2 may be developed, which provides

$$C_2(G) = \frac{\kappa_s \sum_k |f_k|^4 + |\sum_k f_k^2|^2}{(\sum_k |f_k|^2)^2} + 2.$$

We may work out the minimization of C_2, which is not as straight-forward as the minimization of C_1. In the sequel, F will be called *type II* filter, or *type II* filter of parameter $\alpha \in \mathbb{R}$, if exist two distinct integers k_1, k_2 such that $f_k = 0$ for every $k \neq k_1, k_2$, and $f_{k_2} = i\alpha f_{k_1}$.

The following holds

Proposition 1 *For every sub-Gaussian non-circular sequence* $(s_n)_{n\in\mathbb{Z}}$,

$$C_2(G) \geq \frac{\kappa_s}{2} + 2. \quad (5)$$

1. *if* $\kappa_s > -2$, *the equality in (5) occurs if and only if* F *is a type II filter of parameter* 1
2. *if* $\kappa_s = -2$, *the equality in (5) occurs if and only if* F *is a type II filter (any parameter)*

In this contribution, we prove point 2 of Proposition 1, the lack of space preventing from showing point 1, which is quite more involved. Proposition 1 in particular shows that the minimization of C_2 with respect to G "almost" achieves the equalization: if $(s_n)_{n\in\mathbb{Z}}$ is circular, the argument minima of C_2 are such that $z_n = s_n e^{i2\pi\delta f_0 n T_s}$ up to a constant/delay; if $(s_n)_{n\in\mathbb{Z}}$ is real-valued, these minima are such that $z_n = (s_n + \alpha i s_{n-d}) e^{i2\pi\delta f_0 n T_s}$ up to a scale/delay.

[1] except if $\delta f_0 T_s = 0$ $[2\pi]$

Proof of Proposition 1: Inequality 5 holds if and only if

$$(\kappa_s+2)\sum_k |f_k|^4+2\sum_{k\neq l}|f_kf_l|^2(2\cos(2(\theta_k-\theta_l))-\kappa_s)\geq 0 \quad (6)$$

holds, where we set $f_k = |f_k|e^{i\theta_k}$. Consider that $\kappa_s = -2$, hence (6) holds if and only if

$$\sum_{k\neq l}|f_kf_l|^2(\cos(2(\theta_k-\theta_l))+1)\geq 0 \quad (7)$$

which is clearly true. The equality $C_2(G) = 2 + \kappa_s/2$ holds if and only if for all lags $k \neq l$ either $f_kf_l = 0$ or

$$\theta_k - \theta_l = \pi/2\ [\pi]. \quad (8)$$

The end of the proof is by contradiction. Suppose that exist at least three distinct integers k_1, k_2, k_3 such that f_{k_1}, f_{k_2} and f_{k_3} are non-null. We may without restriction assume that $\theta_{k_1} = 0$. The conditions (8) prove that $\theta_{k_2} = \pi/2\ [\pi]$, $\theta_{k_3} = \pi/2\ [\pi]$ and $\theta_{k_3} = \pi/2 + \theta_{k_2}\ [\pi]$. It yields and $\theta_{k_3} = \pi\ [\pi]$ which is impossible. Inequality 5 is an equality if and only if F has two non-null coefficients such that $\theta_{k_2} - \theta_{k_1} = \pi/2\ [\pi]$ hence showing that F is a *type II* filter of whatever parameter.

3 General case of an unknown T_s

This time, we may not consider a sampled version of y at rate $1/T_s$ since T_s is unknown. However, we still stick to the deconvolution ideas of the previous Section. We hence consider any $T > 0$ and the associated time-series $(y_n = y(nT))_{n\in\mathbb{Z}}$ at rate $1/T$. For a given filter G, we still denote by $(z_n)_{n\in\mathbb{Z}}$ the convolution of G and $(y_n)_{n\in\mathbb{Z}}$. Contrary to the case $T = T_s$, the quantities $E|z_n|^2$ and $E|z_n|^4$ vary almost periodically, hence preventing from considering C_2 as a proper cost function. As was suggested in [1], we define a cost function derived from C_2 as[2]

$$C_2(G,T) = \frac{< E|z_n|^4 >}{(< E|z_n|^2 >)^2},$$

where $< E|z_n|^p >$ stands for the 0th order Fourier coefficient of $(E|z_n|^p)_n$, i.e. $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}E|z_n|^p$. The following theorem holds

Theorem 1 *If $(s_n)_{n\in\mathbb{Z}}$ is complex-circular,*

$$C_2(G,T)\geq \kappa_s+2 \quad (9)$$

with equality if and only if $T = T_s$ and G is a filter such that the resulting F is a type I filter.

If $(s_n)_{n\in\mathbb{Z}}$ is not circular,

$$C_2(G,T)\geq \frac{\kappa_s}{2}+2 \quad (10)$$

with equality if and only if G is such that the resulting F is a type II filter.

[2] The abuse of notation is due to the fact that $C_2(G,T_s) = C_2(G)$

Due to the lack of space, we prove the inequality in the non-circular case. We set $\kappa = \kappa_s/2 + 2$. In this respect, notice that Proposition 1 provides the following inequality, true for every time index n: $E|z_n|^4 \geq \kappa(E|z_n|^2)^2$ (this is due to the fact that z_n is a linear mixture of $(s_n)_{n\in\mathbb{Z}}$ times a complex exponential). This gives

$$< E|z_n|^4 > \geq \kappa < (E|z_n|^2)^2 > \quad (11)$$

On the other hand, $< (E|z_n|^2)^2 > \geq (< E|z_n|^2 >)^2$ in virtue of Jensen's inequality. Noticing that $\kappa \geq 0$ (indeed, κ_s is the Kurtosis of $(s_n)_{n\in\mathbb{Z}}$ hence is greater than -2), we finally deduce from Equation (11) that

$$< E|z_n|^4 > \geq \kappa(< E|z_n|^2 >)^2$$

which is the desired result. The same technique applies when $(s_n)_{n\in\mathbb{Z}}$ is circular, by merely setting $\kappa = \kappa_s + 2$ (which is still a positive constant). As far as the cases when Inequalities (9) and (10) are concerned, one may easily adapt the proof sketched in [1].

4 Practical considerations

4.1 Algorithm

This time, $y(t)$ is observed for $t \in [0, T_0]$. As T_0 is finite, the statistics $< E|z_n|^4 >$ and $E|z_n|^2$ are not available, but may be estimated consistently; in this respect, we consider

$$\hat{C}_2(G,T) = \frac{\frac{1}{N}\sum_{n=0}^{N-1}|z_n|^4}{(\frac{1}{N}\sum_{n=0}^{N-1}|z_n|^2)^2}$$

which is a consistent estimate of $C_2(G,T)$. It can be shown that $T \mapsto C_2(G,T)$ is constant except on a countable set. This remark, and the fact that we cannot give explicitly the dependency of C_2 in T justifies our optimization procedure:

- for every T of a grid, compute the available samples $y_n = y(nT)$
- from these samples, perform the minimization of $G \mapsto \hat{C}_2(G,T)$ using a Newton method
- Estimate T_s as the T of the grid for which $\inf_G \hat{C}_2(G,T)$ is minimum.

5 Simulation results

We fix the excess bandwidth factor to 0.2. The propagation channels result from the superposition of 3 paths: the delays are uniformly chosen as random between 0 and $3T_s$, the directions of arrival are uniformly distributed in $[0, 2\pi]$. The attenuation are uniformly distributed in $[0, 1]$. Two sensors are used. The filter G has four taps. The observation duration is $T_0 = 1000T_s$.

We first consider a circular modulation: for instance a QAM16. For a frequency offset of $\delta f_0 = \frac{0.05}{T_s}$, the values of $\inf_G \hat{C}_2(G,T)$ are displayed on the left part of Figure 1. Notice that, as the theory predicts, the minimum over the parameters T is $1.32 \simeq \kappa_s + 2$, this minimum being reached for $\hat{T}_s \simeq T_s$.The outputs z_n of the filter G for which the minimum of $\hat{C}_2(G, \hat{T}_s)$ is reached are plotted on the right part of Figure 1. Of course, we could have worked out the complete equalization (estimation of $\delta f_0 T_s$, compensation of the frequency offset, and estimation of the symbols).

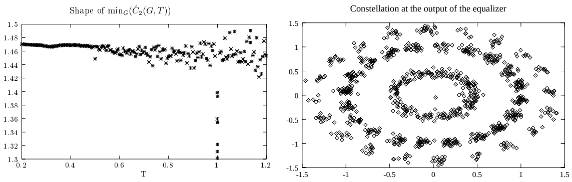

Figure 1: QAM16 - frequency offset $\delta f_0 T_s = 0.05$

We now consider a non circular modulation: the symbols are $BPSK$: hence the Kurtosis of $(s_n)_{n\in\mathbb{Z}}$ is $\kappa_s = -2$ For a frequency offset of $\delta f_0 = \frac{0.05}{T_s}$, the values of $\inf_G \hat{C}_2(G, T)$ are displayed on the left part of Figure 2. Notice that, as the theory predicts, the minimum over the parameters T is close to $\kappa_s/2 + 2 = 1$, again, this minimum being reached for $\hat{T}_s \simeq T_s$. The outputs z_n of the filter G for which the minimum of $\hat{C}_2(G, \hat{T}_s)$ is reached are plotted on the right part of Figure 2.

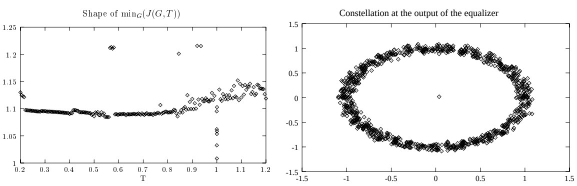

Figure 2: PSK2 - frequency offset $\delta f_0 T_s = 0.05$

We now illustrate the robustness of our approach to the frequency offset for different levels of Signal-to-Noise Ratio (SNR) and different modulations. The noise is assumed to be complex Gaussian, temporally and spatially white. We compare the probability of correct detection of the period symbol of our method when the frequency offset is equal to zero and when it is chosen random, uniformly distributed in $[0, 1/4]$. By "correct detection", we mean that $\frac{|\hat{T}_s - T_s|}{T_s} < \frac{1}{1000}$. 200 realizations of the channel/frequency offset are run. We first consider the case of a circular modulations, namely PSK4 and PSK16 modulations are considered. The results are presented in Table 1. By "standard", we mean the standard second-order cyclic approach (actually the optimally weighted version): see [3][2].

We deduce that the estimation of T_s is not affected by the frequency offset and, more strikingly, that our estimate outperforms the standard estimate.

We also test two non-circular modulations: the BPSK and the PAM4. Results are presented in Table 2. Once

Mod:	PSK4		PSK16	
	$\delta f_0 \neq 0$	$\delta f_0 = 0$	$\delta f_0 \neq 0$	$\delta f_0 = 0$
5dB (deconvo.)	95	98.5	91	95
5dB (standard)	45	44	42	43.5
10dB (deconvo.)	100	100	99	98
10dB (standard)	68	68.5	65	66.5
20dB (deconvo.)	100	99	99	99
20dB (standard)	77	77.5	76	76

Table 1: Proba. of correct detection - circular modul.

again, the superiority of the performance of our approach is noticed. The frequency offset does perturb significantly our estimate of the symbol period, except when the SNR is very low.

	BPSK		PAM4	
	$\delta f_0 \neq 0$	$\delta f_0 = 0$	$\delta f_0 \neq 0$	$\delta f_0 = 0$
5dB (deconvo.)	78	98	68	89
5dB (standard)	51	52	50	49.5
10dB (deconvo.)	96	99	95	97
10dB (standard)	71.5	72	74	75
20dB (deconvo.)	98	99	96	99
20dB (standard)	83.5	83	82	82.5

Table 2: Proba. of correct detect.: the non-circular case

6 Conclusion

In this contribution, we develop a methodology for the estimation of the symbol period when an unknown frequency offset affects the observation. It relies on the idea of deconvolution. We show that our method allows to estimate the symbol period. The algorithm has been validated by extensive simulations which confirm its superiority over the existing methods when the excess bandwidth is small.

References

[1] S. Houcke, A. Chevreuil, and P. Loubaton, "Joint blind equalization and estimation of the symbol period: a contrast function approach.," in *Proc. of ICASSP*, Salt Lake City (USA), 2001.

[2] F. Gini and G.B. Giannakis, "Frequency offset and symbol timing recovery in flat-fading channels: a cyclostationary approach," *IEEE Trans. on Comm.*, vol. 46, no. 3, pp. 400–411, march 1998.

[3] L. Mazet and P. Loubaton, "Cyclic correlation based symbol rate estimation," in *Proceedings of ASILOMAR*, Pacific Grove, California, october 1999, pp. 1008–1012.

[4] L. Izzo and A. Napolitano, "Higher-order cyclostationarity properties of sampled time-series," *Signal Processing*, vol. 54, no. 3, pp. 303–307, november 1996.

[5] O. Shalvi and E. Weinstein, "New criteria for blind deconvolution of nonminimum phase systems (channels)," *IEEE Trans. on Information theory*, vol. 36, no. 2, pp. 312–321, March 1990.