Behavior and Corrections of Constant Modulus Equalization with a DC offset

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ABSTRACT
Some communication systems embed pilot tones into the data spectrum to aid the receiver in synchronization, which may result in a DC offset of the baseband data. This paper motivates the use for adaptive estimation of the DC offset at the receiver, and proposes a Constant Modulus cost function that is jointly minimized over the DC offset estimate and the equalizer parameters. It is shown that for an arbitrary DC offset, the CM cost function admits local spurious minima in terms of equalizer settings, and that matching the DC offset at the receiver to the level inserted at the transmitter causes both local and global minima in terms of equalizer settings. Extrema of the cost function in terms of a DC offset are derived and classified, and adaptive methods for DC offset estimation are introduced.

1 Introduction

To achieve the low error rates necessary for high quality of services in modern digital communications systems, it is common to embed training or pilot signals into the data packets or data spectrum to aid the receiver in equalization and/or in synchronization. For example, the 10 Megabaud terrestrial broadcast of Digital Television (DTV) signals in the United States uses a single-carrier, single sideband modulation scheme (known as 8-VSB for 8-level Vestigial Sideband Modulation) that contains a bi-level training sequence inserted at the header of each data frame, occuring every 24ms, to aid in adaptive equalization. Furthermore, a narrowband pilot tone is inserted into the lower band edge of the 8-VSB data spectrum, containing about 7.5 percent of the power of the data spectrum, to aid in carrier synchronization [1]. Unfortunately, the training sequence is both too short and too infrequent to be reliable, and the pilot tone may do more harm than good if not properly processed.

Our work studies the effect of a DC offset on equalization due, for example, to pilot tone insertion, and proposes blind methods to estimate and remove the DC offset by jointly minimizing a Constant Modulus (CM) cost function [2] over equalizer parameters and a DC offset estimate. Our results can be described by the following three observations. First, we show that with an arbitrary DC offset the CM cost function admits local spurious minima in terms of the equalizer setting. Second, if the DC offset at the receiver is assumed to be equal to the DC offset inserted at the transmitter, which is equivalent to neglecting the ISI caused by the channel, then only half of the CM equalizer minima remain unchanged, i.e., as if the system has no DC offset. Moreover, these global minima lead to source estimation with polarity that can be predicted by the sign of the DC offset. Third, provided a power constraint on the signal space is satisfied, the DC offset estimate, computed with a joint optimization of the CM cost function, is guaranteed to converge to the desired solution, and the equalizer parameters are guaranteed to converge to the desired solution.

Though we are immediately interested in application to DTV signals, the sequel is generalized to an arbitrary DC offset inserted at the transmitter and is organized as follows. §2 describes the data model and CM receiver basics. §3 compares the effects of the DC offset on CM and Mean Square Error (MSE) receivers. §4 contains an analysis of the joint CM optimization over DC offset estimation and equalizer settings. Section 5 contains simulation results. Concluding remarks are contained in §6.

2 Model

2.1 Data model

The output $y(n)$ of a combined channel-equalizer system is described by the equation,

$$y(n) = \underline{h}^t \underline{s}(n) + \underline{f}^t \underline{w}(n). \tag{1}$$

The l-th component of the vector source $\underline{s}(n)$ is denoted $a_l(n) + p$. The vector $\underline{s}(n)$ contains a snapshot of symbols representing the signal of interest coming from a single source or multiple sources. The sequence $\{a_l(n)\}_{n \in \mathbb{Z}}$ of source symbols defines a zero-mean, sub-gaussian stochastic process. We denote $\mathbb{E}\{a^2\}$ as the variance of the unbiased source signal. The constant DC offset p added to the source symbols is, for example, a pilot signal introduced to help the receiver in carrier synchronization. Additive perturbations on the channel are included in the noise vector $\underline{w}(n)$ of dimension N. The noise $\underline{w}(n)$ is assumed to be Gaussian, temporally and spatially white and zero-mean with variance $\mathbb{E}\{w^2\}$. The noise contribution is filtered by the linear receiver $\underline{f}$ of same dimension. The vector $\underline{h}$ denotes the combined finite impulse response of the channel-receiver system. We assume that $\underline{h}$ and $\underline{s}(n)$ are of length M. Finally, we assume that all quantities are real-valued.

2.2 CM receiver

The CM equalizers are denoted by $\underline{f}^*$ and defined as the minima of the criterion,

$$J_c^{(dc)}(\underline{f}, \hat{p}) = \mathbb{E}\left\{ \left((y(n) - \hat{p})^2 - \gamma \right)^2 \right\}. \tag{2}$$

The parameter $\hat{p}$ is introduced in the receiver to compensate for the DC component p that is inserted at the transmitter.

The local minima $\hat{p}$ of $J_c^{(dc)}$ are denoted $\hat{p}_*$. The so-called dispersion constant of the CM criterion is denoted by γ. The purpose of this paper is to describe the effect of the DC offset on the CM equalizers and to compare these solutions to the CM equalizers when the source is unbiased.

3 Effect of DC offset on equalizer settings

3.1 Observations with regards to CM receivers

When there is no DC offset inserted at the transmitter, in the absence of noise and under the assumption that all global impulse responses $\underline{h}$ are reachable, the spike vectors $\underline{h}^* = \pm \underline{e}_k$ (for which the non-zero component $+1$ is at the $(k+1)-$th position) are global minima of the CM cost function with $\gamma = \frac{E\{a^4\}}{E\{a^2\}}$, [3][4]. However, when the transmitter inserts a DC offset and the source is biased as in model (1), then the minima settings of the CM cost function (2) depend on the DC offset p introduced at the source. In particular, if the DC offset estimate of the received signal $\hat{p}$ is not properly chosen, then the CM criterion admits local spurious minima. This observation is described in Proposition 1.

Proposition 1. *With* $\gamma = \frac{E\{a^4\}}{E\{a^2\}}$*, for a given* $\hat{p} = p$*, only the vectors* $\underline{h}^* = sign(p)\,\underline{e}_k$ *for* $k \in \{0, \cdots, M-1\}$ *are global minima of the criterion* $J_c^{(dc)}(\underline{h}, \hat{p})$*. The vectors* $\underline{h}^* = -\,sign(p)\,\underline{e}_k$ *are no longer global minima of the criterion* $J_c^{(dc)}(\underline{h}, \hat{p})$ *as they are for the CM criterion when the source is unbiased.*

Proposition 1 describes what is happening to the minima of the CM criterion if the DC offset p introduced at the source is removed at the output of the CM receiver when that offset is known by the receiver. By selecting $\hat{p} = p$, this correction of the bias, which seems intuitively reasonable, leads to a surprising result. Only half of the global minima of the CM criterion for an unbiased source remain global minima of the criterion when the source is biased and when such a correction is applied. The polarity of the global minima is decided by the sign of the DC offset at the transmitter.

Figure 1 further illustrates Proposition 1 by showing the cost surface of the criterion (2) for a two-tap global impulse response $\underline{h}$ and BPSK signaling ($a(n) = \pm 1$) with $\hat{p} = p = 0.25$. The criterion admits a pair of desirable global minima $(0,1)$ and $(1,0)$. However, the minima at negative polarities have moved away from spike vectors. These local minima have contracted to approximately $(-0.7, 0.1)$ and $(0.1, -0.7)$. Notice that selecting $\hat{p} = p$ is not necessarily a good choice. In general, criterion (2) admits local spurious minima with respect to the equalizer settings.

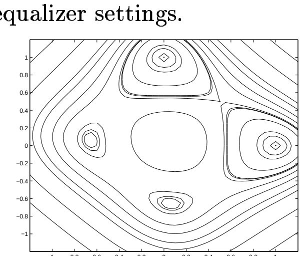

Figure 1: Contour of CM cost function, $\hat{p} = p = 0.25$.

3.2 Comparison with MSE settings

Next we compare the results described in Proposition 1 for the CM cost function with a MSE cost function. A MSE cost function for the biased signal is given by

$$J_{mse}^{(dc)}(\underline{f}, \hat{p}) = E\left\{(y(n) - \hat{p} - a(n-d))^2\right\} \qquad (3)$$

where d is a delay between 0 and $M-1$. In the absence of noise, minimization of the MSE criterion with respect to the equalizer settings admits a global impulse response of the form

$$h_l = \begin{cases} \omega & \text{for } l \neq d \\ 1+\omega & \text{for } l = d \end{cases} \qquad (4)$$

where $\omega = \frac{\hat{p}p - Rp^2}{E\{a^2\}}$, $R = \frac{M\hat{p}p + E\{a^2\}}{Mp^2 + E\{a^2\}}$.

Notice that for an arbitrary choice of p and $\hat{p}$ the global minimum of the MSE criterion is biased by the factor ω which is a function of p and $\hat{p}$. However, for the specific case of $\hat{p} = p$ we have $\omega = 0$, which is precisely the solution that leads to a perfect source estimation. This observation is in contrast to our results for the CM criterion as illustrated in the previous subsection. Notice that the result $\omega = 0$ can be achieved as well with the choice $\hat{p} = E\{y\} = p \sum_l h_l$.

4 DC offset estimation

In this section we describe simple solutions for the estimation of the optimum DC correction $\hat{p}$. To understand how the DC offset affects the CM criterion, we first rewrite the cost function $J_c^{(dc)}(\underline{f}, \hat{p})$ as follows:

$$J_c^{(dc)}(\underline{f}, \hat{p}) = J_c(\underline{f}, 0) + \mathcal{E}^4 + 6\mathcal{E}^2\left(E\{z^2\} - \frac{\gamma}{3}\right) \qquad (5)$$

where $\mathcal{E} = \hat{p} - E\{y\}$ and it is understood that $J_c(\underline{f}, 0) = E\{\left((\underline{h}^t\underline{a}(n) + \underline{f}^t\underline{w}(n))^2 - \gamma\right)^2\}$ is the unbiased CM cost function. The terms $\mathcal{E}^2$ and $\mathcal{E}^4$ can be seen as additive constraints on the unbiased CM criterion. The term $E\{z^2\}$ denotes the power of the receiver output of the unbiased signal defined as $z(n) = \underline{h}^t\underline{a}(n) + \underline{f}^t\underline{w}(n)$.

Estimation of the DC offset can be deduced by an optimization of the cost function $J_c^{(dc)}(\underline{f}, \hat{p})$ with respect to the parameter $\hat{p}$. Next, we furnish a description of the minima $\hat{p}_*$ of the cost function $J_c^{(dc)}(\underline{f}, \hat{p})$, simple methods for the estimation of the optimal DC offset, and conditions for global convergence of the algorithms.

4.1 Offset minima of the CM cost function

The extrema of (5) with respect to the DC correction term $\hat{p}$ are given in Proposition 2.

Proposition 2. *The extrema* $\hat{p}_*$ *of* $J_c^{(dc)}(\underline{f}, \hat{p})$ *are given by,*

1. $\hat{p}_* = E\{y\}$
2. $\hat{p}_* = E\{y\} \pm \sqrt{\gamma - 3E\{z^2\}}$

If $(E\{z^2\} - \frac{\gamma}{3})$ *is positive, then solution 1 corresponds to a minimum and solution 2 does not exist. If* $(E\{z^2\} - \frac{\gamma}{3})$ *is negative, then solution 1 corresponds to a global maximum and solution 2 describes two global minima.*

For the solution $\hat{p}_* = E\{y\}$, the joint cost function $J_c^{(dc)}(\underline{f}, \hat{p})$ is equivalent to the CM cost function with an unbiased source. Therefore, no local spurious minima for the equalizer settings are introduced. Notice that this solution corresponds to the case where the power constraint $(E\{z^2\} - \frac{\gamma}{3})$ is positive and describes a region where CM equalizer extrema settings are located for an unbiased source [5][6]. In the next subsection we describe adaptive algorithms that provide an estimation of this desired solution.

4.2 Adaptive algorithms for DC correction

The DC correction term can either be calculated directly with an empirical estimator of $E\{y\}$, or deduced as a stationary point of a stochastic gradient algorithm minimizing

$J_c^{(dc)}(f, \hat{p})$ over $\hat{p}$. The first approach admits a simple online calculation of the mean of the equalizer output by use of a leaky integrator as follows,

$$\hat{p}(n+1) = \hat{p}(n) + \alpha(y(n) - \hat{p}(n)) \tag{6}$$

where usually α is small. Alternatively, a gradient optimization technique of the CM criterion leads to an adaptive algorithm given by,

$$\hat{p}(n+1) = \hat{p}(n) + \epsilon\mu\, \mathcal{E}_y \tag{7}$$

where $\mathcal{E}_y = \left((y(n) - \hat{p}(n))^2 - \gamma\right)(y(n) - \hat{p}(n))$ and where $\epsilon = -sign\left(\mathbb{E}\{z^2\} - \frac{\gamma}{3}\right)$. The scalar μ is a positive number, usually small, known as the step-size of the algorithm.

Observe that there is little difference in numerical computational complexity between the two approaches. The term $\mathcal{E}_y$ in equation (7) need already be computed for the adaptation of the CM equalizer taps, and is therefore available for the adaptation of the DC correction term in (7). However, the polarity indicator ϵ needs to be calculated to evaluate the sign of the power constraint and guarantee convergence to the desired solution $\hat{p}_* = \mathbb{E}\{y\}$.

If the polarity indicator ϵ is not calculated, to save computations, then we need to identify the extrema of the joint cost function $J_c^{(dc)}(\underline{f}, \hat{p})$ with respect to the vector $\underline{f}$, to verify the non-existence of local spurious minima. Proposition 3 identifies and classifies these extrema.

Proposition 3. *Assuming that all vectors $\underline{h}$ are achievable in the absence of noise, the joint optimization of the criterion $J_c^{(dc)}(\underline{f}, \hat{p})$ leads to channel-equalizer extrema $\underline{h}^*$ of the form,*

$$\underline{h}^* = h^* \sum_{k \in \nu} \underline{e}_k \tag{8}$$

where ν denotes the set of indices associated to non-zero components of the vector $\underline{h}^*$. The cardinality of ν, i.e., the number of non-zero components of $\underline{h}^*$ is denoted by $|\nu|$, and h^* is a scalar. The extrema (8) are classified as follows:

1. *For $\hat{p}_* = \mathbb{E}\{y\}$, we have*
   - *$\underline{h}^* = \underline{0}$ (maximum),*
   - *$\underline{h}^* = \pm\sqrt{\gamma \frac{\mathbb{E}\{a^2\}}{\mathbb{E}\{a^4\}}}\, \underline{e}_k$ for $k \in \{0, \cdots, M-1\}$ (global minima),*
   - *$\underline{h}^* = \pm\sqrt{\frac{\gamma}{\sigma_a^2}} \sqrt{\frac{1}{\kappa_a - 3 + 3|\nu|}} \sum_{k \in \nu} \underline{e}_k$ with $|\nu| > 1$ (saddle points).*
2. *For $\hat{p}_* = \mathbb{E}\{y\} \pm \sqrt{\gamma - 3\mathbb{E}\{z^2\}}$ we have,*
   - *$\underline{h}^* = \underline{0}$ (minimum),*
   - *$\underline{h}^* = \pm\sqrt{\gamma \frac{\mathbb{E}\{a^2\}}{\mathbb{E}\{a^4\}}} \sqrt{\frac{2\kappa_a}{9 - \kappa_a}}\, \underline{e}_k$ for $k \in \{0, \cdots, M-1\}$ (global maxima)*
   - *$\underline{h}^* = \pm\sqrt{\frac{\gamma}{\sigma_a^2}}, \sqrt{\frac{-2}{\kappa_a - 3 - 6|\nu|}} \sum_{k \in \nu} \underline{e}_k$ with $|\nu| > 1$ (saddle points)*

where $\kappa_a = \frac{\mathbb{E}\{a^4\}}{\mathbb{E}\{a^2\}^2}$ is the kurtosis of the unbiased source.

Proof: See Appendix.

Proposition 3 classifies extrema of the equalizer settings associated with each of the DC correction solutions furnished in Proposition 2. At the solution $\hat{p}_* = \mathbb{E}\{y\}$, the joint CM cost function is equivalent to the CM cost function with an unbiased source. In the absence of noise, and for a perfectly invertible channel, the minima of the joint cost function therefore provide perfect source recovery. Thus when the power constraint $(\mathbb{E}\{z^2\} - \frac{\gamma}{3})$ is positive, the stochastic optimization of the joint CM criterion is guaranteed to converge to desirable equalizer settings and DC offset.

At the solution $\hat{p}_* = \mathbb{E}\{y\} \pm \sqrt{\gamma - 3\mathbb{E}\{z^2\}}$, desirable equalizer settings which furnish a perfect estimation of the source are defined as the global maxima of the joint CM criterion. These global maxima can be achieved by gradient ascent, rather than gradient descent. However, the choice of gradient ascent or descent of equalizer coefficient adaptation also requires the calculation of a polarity indicator ϵ. Effectively, the issue of the polarity selection has been moved from DC correction adaptation to equalizer parameter adaptation.

5 Simulations

An example of DC offset estimation is given in Figure 2. The flat dashed line is the value of $p = 1.25$ inserted at the transmitter. Line #2 corresponds to the estimation given by the leaky integrator in equation (6). Line #3 corresponds to the estimation by the stochastic gradient descent in equation (7). Algorithms (6) and (7) both use the same parameter $\alpha = \mu = 0.01$. The channel taps are given as $(1, -0.6, 0.3, 0.1)$, and $N = 16$ is the length of the equalizer with BPSK signaling. Observe that the estimation furnished by the stochastic gradient in equation (7) tracks well the optimum estimate corresponding to line #1. Both algorithms converge near the flat line corresponding to the DC offset as the eye is open(ed). The residual error is due in this case to a lack of perfect invertibility of the channel with a finite length equalizer. In practice either estimation method is applicable. The jitter of the leaky integrator can be reduced by reducing α.

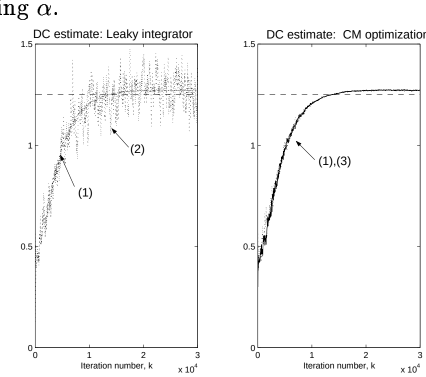

Figure 2: DC offset estimate. Comparison between the empirical mean, exact mean and gradient stochastic.

6 Conclusion

This paper has motivated the use for adaptive estimation of DC offset in a communication system that has a DC offset in the data spectrum, inserted at the transmitter. A Constant Modulus cost function was proposed and analyzed, which is jointly minimized over DC offset estimate and equalizer parameters. Adaptive methods were introduced, and computer simulations illustrate our results.

Appendix: Proof of Proposition 3

1. Extrema characterization

The CM cost function optimized over $\hat{p}$ can be written as, $J_c^{(dc)}(\underline{f}, \hat{p}_*) = J_c(\underline{f}, 0) - \lambda\left(\mathbb{E}\{z^2\} - \frac{\gamma}{3}\right)^2$, where λ is given by $\lambda = 0$ for the solution 1 (i.e., $\hat{p}_* = \mathbb{E}\{y\}$) and $\lambda = 9$ for the solution 2 (i.e., $\hat{p}_* = \mathbb{E}\{y\} \pm \sqrt{\gamma - 3\mathbb{E}\{z^2\}}$). In the absence of noise, the criterion can be rewritten in terms of the global impulse response $\underline{h}$,

$$J_c^{(dc)}(\underline{h},\hat{p}_*) = (\kappa_a-3)\mathbb{E}\{a^4\}\|\underline{h}\|_4^4 + 3\mathbb{E}\{a^2\}^2\,\|\underline{h}\|_2^4 \quad (9)$$
$$-2\gamma\mathbb{E}\{a^2\}\|\underline{h}\|_2^2 - \lambda(\mathbb{E}\{a^2\}\|\underline{h}\|_2^2 - \frac{\gamma}{3})^2 + \gamma^2.$$

The extrema $\underline{h}^*$ are the solutions zeroing the gradient with respect to $\hat{p}$ of the criterion $J_c^{(dc)}$. A straightforward calculation leads to the system of equations

$$4(\kappa_a-3)\mathbb{E}\{a^4\}(h_k^*)^3 + 4Q\mathbb{E}\{a^2\}^2(3-\lambda)h_k^*$$
$$-4\gamma(\mathbb{E}\{a^2\}-\frac{\gamma}{3})^2 h_k^* = 0 \quad (10)$$

for $k=0,\cdots,M-1$, where $Q=\sum_k (h_k^*)^2$. If all global impulse responses $\underline{h}$ can be reached, the system of equations above is not coupled. Under this assumption, the extrema of the biased CM criterion are given by

$$(h_k^*)^2 = \begin{cases} 0 & \text{or} \\ (1-\frac{\lambda}{3})\frac{\gamma-3Q\sigma_a^2}{\sigma_a^2(\kappa_a-3)}. \end{cases} \quad (11)$$

We can use the equation above to compute the norm of $\underline{h}^*$ defined by $Q=\sum_k (h_k^*)^2$. We get,

$$Q=(1-\frac{\lambda}{3})\frac{P\gamma}{\kappa_a+3(P-1)-\lambda P} \quad (12)$$

where $P\geq 1$ denotes the number of non-zero components. If we plug this equation into (11), we get the expression of the components $(h_k^*)^2$ as a function of P and the statistics of the source. We have,

$$(h_k^*)^2=(h^*)^2=(1-\frac{\lambda}{3})\frac{\gamma}{\sigma_a^2(\kappa_a+3(P-1)-\lambda P)} \quad (13)$$

The extrema can therefore be classified into three categories,

- $\underline{h}^*=\underline{0}$ when $P=0$
- $\underline{h}^*=\frac{\gamma}{\kappa_a\sigma_a^2}\left(\frac{1-\lambda/3}{1-\lambda/\kappa_a}\right)\underline{e}_k$ when $P=1$
- $\underline{h}^*=(1-\frac{\lambda}{3})\frac{\gamma}{\sigma_a^2(\kappa_a+3(P-1)-\lambda P)}\sum_{k\in\nu}\underline{e}_k$ when $P>1$

To classify the extrema we need to characterize the sign of the quadratic form $\underline{x}^t H(\underline{h}^*)\underline{x}$, where $H(\underline{h}^*)$ denotes the Hessian of $J_c^{(dc)}(\underline{f},\hat{p}_*)$ on each setting.

2. Stability of the extrema

The (j,k) component of the Hessian matrix $H(\underline{h}^*)$ is given by,

$$\frac{\partial^2 J_c^{(dc)}}{\partial h_j\partial h_k} = \begin{cases} \pm 8\,h_j^* h_k^*\,\sigma_a^2(3-\lambda) & \text{for } j\neq k \\ 12(\kappa_a-3)\sigma_a^4(h^*)^2+8\sigma_a^4(3-\lambda)(h^*)^2+ & \\ 4N\sigma_a^4(3-\lambda)-4\gamma\sigma_a^2(1-\lambda/3) & \text{for } j=k \end{cases} \quad (14)$$

Using (12) and (13) in the previous equation, the result can be simplified to

$$\frac{\partial^2 J_c^{(dc)}}{\partial h_j\partial h_k}=\frac{8}{3}\frac{\gamma(3-\lambda)}{\kappa_a+3(P-1)-\lambda P}\begin{cases} \pm(3-\lambda),\text{or } 0 & \text{for } j\neq k \\ \sigma_a^2(\kappa_a-\lambda) & \text{for } j=k \end{cases} \quad (15)$$

when $P\neq 0$, and

$$\frac{\partial^2 J_c^{(dc)}}{\partial h_j\partial h_k} = \begin{cases} 0 & \text{for } j\neq k \\ -\frac{4}{3}\gamma\sigma_a^2(3-\lambda) & \text{for } j=k \end{cases} \quad (16)$$

when $P=0$.

2.1 Stability of the solutions $\underline{h}^*=\underline{0}$

The Hessian is a diagonal matrix defined by $H(\underline{0}) = -\frac{4}{3}\gamma\sigma_a^2(3-\lambda)I_M$. For $\lambda=0$, the quadratic form $\underline{x}^t H(\underline{0})\underline{x} = -4\gamma\sigma_a^2\|\underline{x}\|^2<0$ is always negative. The extremum $\underline{h}^*=\underline{0}$ is thus a maximum. The result is inverted when $\lambda=9$. Indeed in this case the quadractic form $\underline{x}^t H(\underline{0})\underline{x}=8\gamma\sigma_a^2\|\underline{x}\|^2>0$ is always positive. The extremum $\underline{h}^*=\underline{0}$ is thus a minimum.

2.2 Stability of the solutions $\underline{h}^*=h^*\,\underline{e}_k$

The Hessian matrix is of the form $H(h^*\,\underline{e}_k) = \frac{8}{3}\frac{\gamma\sigma_a^2(3-\lambda)(\kappa_a-\lambda)}{\kappa_a+3(P-1)-\lambda P}I_M$. For $\lambda=0$, we have $\underline{x}^t H(h^*\,\underline{e}_k)\underline{x} = 8\gamma\sigma_a^2\|\underline{x}\|^2>0$. The spike vectors $\underline{h}^*=h^*\,\underline{e}_k$ define therefore a set of minima. For $\lambda=9$, we get $\underline{x}^t H(h^*\,\underline{e}_k)\underline{x} = -16\gamma\sigma_a^2\|\underline{x}\|^2<0$. In this case, the vectors $\underline{h}^*=h^*\,\underline{e}_k$ define a set of maxima.

2.3 Instability of the other extrema

When $\underline{h}^*\neq h^*\,\underline{e}_k$ and $\underline{h}^*\neq 0$ the quadratic form $\underline{x}^t H(\underline{h}^*)\underline{x}$ can be expressed as follows,

$$\underline{x}^t H(\underline{h}^*)\underline{x}=\alpha\sum_i x_i^2+\sum_{k,l\neq i}(x_k,x_l)\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}(x_k,x_l)^t \quad (17)$$

where the two summations are disjoint. We have $\alpha = \frac{8}{3}\frac{\gamma\sigma_a^2(3-\lambda)(\kappa_a-\lambda)}{\kappa_a+3(P-1)-\lambda P}$ and $\beta=\pm\frac{8}{3}\frac{\gamma(3-\lambda)^2}{\kappa_a+3(P-1)-\lambda P}$. The sign of the quadratic form depends therefore on the sign of α and the sign of the eigenvalues of the 2×2 matrix introduced in the expression (17). Indeed, there exists a vector $\underline{x}_1$ such that $\underline{x}_1^t H(\underline{h}^*)\underline{x}_1 = \alpha\sum_i x_i^2$, where $\alpha = \frac{-16\gamma\sigma_a^2(\kappa_a-9)}{\kappa_a-6-6P}<0$ when $\lambda=9$ and $\alpha=\frac{8\gamma\sigma_a^2\kappa_a}{\kappa_a-3+3P}>0$ when $\lambda=0$. In the same way, there exists a vector $\underline{x}_2$ such that $\underline{x}_2^t H(\underline{h}^*)\underline{x}_2 = \alpha\sum_{k,l}(x_k^2+x_l^2)+2\beta\sum_{k,l}x_k x_l$. To determine the sign of the eigenvalues of the second order quadratic form, it is equivalent to study the sign of the determinant $\alpha^2-\beta^2$. We have,

$$\alpha^2-\beta^2=(\frac{8}{3})^2\frac{\sigma_a^4(3-\lambda)^2}{(\kappa_a+3(P-1)-\lambda P)^2}(\kappa_a-3)(\kappa_a+3-2\lambda) \quad (18)$$

Thus for $\lambda=0$, since $\kappa_a-3<0$ we have $\alpha^2-\beta^2<0$. For $\lambda=9$, we have $\alpha^2-\beta^2>0$. The sign of the quadratic form $\underline{x}^t H(\underline{h}^*)\underline{x}$ changes when $\underline{x}=\underline{x}_1$ or $\underline{x}=\underline{x}_2$. The extrema $\underline{h}^*$ with more than one non-zero component correspond therefore to saddle points of $J_c^{(dc)}(\underline{f},\hat{p}_*)$.

2. Classification of the extrema $(\underline{h}^*,\hat{p}^*)$

For $\lambda=0$ the extrema $\underline{h}^*\neq\underline{0}$ satisfy $\|\underline{h}^*\|^2\geq\frac{\gamma}{3\mathbb{E}\{a^2\}}$ and $\|\underline{h}^*\|^2<\frac{\gamma}{3\mathbb{E}\{a^2\}}\frac{1}{1-\frac{1}{P}}$ for $P>1$. In other words, we have $\mathbb{E}\{\underline{y}_*{}^2\}-\frac{\gamma}{3}>0$ which implies that the solution $\hat{p}_*=\mathbb{E}\{y\}$ is a minimum. For $\lambda=9$ the extrema $\underline{h}^*\neq\underline{0}$ satisfy $\|\underline{h}^*\|^2\leq\frac{\gamma}{3\mathbb{E}\{a^2\}}$ and $\|\underline{h}^*\|^2>\frac{\gamma}{3\mathbb{E}\{a^2\}}\frac{1}{1+\frac{1}{2P}}$ for $P>1$. Therefore we have $\mathbb{E}\{\underline{y}^2\}-\frac{\gamma}{3}<0$ which implies that the solution $\hat{p}_*=\mathbb{E}\{y\}\pm\sqrt{\gamma-3\mathbb{E}\{z^2\}}$ is a maximum.

□□□

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