AN EXACT DISCRETE BACKPROJECTION OPERATOR

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ABSTRACT

This paper presents a new version of the exact discrete Radon [1] transform denoted as Mojette transform. Whereas the Mojette transform backprojects a single bin into a grey value pixel at each iteration, the proposed version backprojects a bin onto an entire segment of binary pixels. Implementation of this discrete exact backprojector is presented and used for specific binary images of incomplete contours giving a new representation of 2D edges images.

1 INTRODUCTION

Incomplete contours coding is still an open area. Much of the classical methods for contours (Freeman, RLE, Fourier descriptors) need complete contours to be efficient. In other words, the goal of this paper is to develop a new transform able to reduce the entropy of the 2D source. This is elegantly done here using a discrete projection method. The binary image is projected onto 1D integers bins. Thanks to an a priori knowledge of an unbalanced binary density function, only few projections suffice to reconstruct the entire data set, therefore the method compresses the initial data. Section 2 recalls the main principles of the Mojette transform and section 3 presents both the discrete backprojector algorithm and its complexity properties. Section 4 discusses the choice of the projections set and give a second algorithm for choosing the minimum set. Finally section 5 exemplifies the approach onto both real contours images and for bitstreams mapped onto a 2D geometrical buffer.

2 THE MOJETTE TRANSFORM

The Mojette transform has been described and used for the last five years [2]. It corresponds to a linear discrete exact Radon transform, i.e. a set of discrete projections describing a discrete image, a convex region or a volume. Projection angles are chosen among discrete directions $\theta = atan(p, q)$ where p and q are integers prime together ($GCD(p, q) = 1$). Thus the equation of the transform is given in 2D by :

$$Mf(k,l) = \{M_i f(k,l) = proj(p_i, q_i, m); i \in I\}, \quad (1)$$

with

$$M_i f(k,l) = \sum_k \sum_l f(k,l)\Delta(m - p_i k + q_i l), \quad (2)$$

and where $\Delta(m)$ is the discrete Kronecker function. The transform is thus linear both in the number of projections I and the number of pixels denoted by N. The number of points, denoted as bins in the following, onto a projection of direction (p, q) depends upon the shape of the support. This linear transform generates redundancy usually computed with the following *red* index :

$$red = \frac{\#bins}{\#pixels} - 1. \quad (3)$$

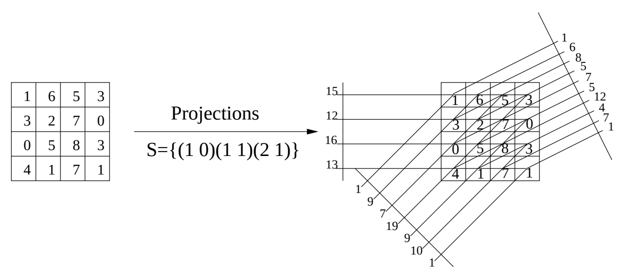

Figure 1: Exemple of a projection set of a binary shape

2.1 Algorithms

The direct and inverse transform algorithms are now presented. They both have a complexity order of $O(IN)$.

2.1.1 Direct Algorithm

```
for indexproj ← 1 to I
  for indexpixel ← 1 to N
   b ← p.lig-q.col
   bin(p,q,b) ← +image(lig,col)
  endfor
endfor
```

2.1.2 *Inverse Algorithm*

The inverse transformation can be implemented with an algorithm that has the same complexity than the direct transform, i.e. $O(IN)$ with only (integer or modulo) additions/subtractions operators instead of finding an inverse matrix formulation.

```
for indexpixel ← 1 to N
  (n,b) ← Step1
  (lig,col) ← Step2(n,b)
  image(lig,col) ← bin(p,q,b)
  for indexprojection ← 1 to I
   update corresponding bin
  endfor
endfor
```

Step 1 List of one-to-one correspondence between bins and pixels

Step 2 Linking information between bins and pixels allows to easily retrieve pixel coordinates

3 THE BACKPROJECTION OPERATOR

The classical Mojette transform as described above applies to convex shapes. In this section a new version of this transform is presented. This new version has been designed to allow projection data description when non-convex shapes are taken into account. Another way to start with is considering supports with an a priori knowledge different than the usual support convexity. This new version of the Mojette transform backprojects a segment instead of a single bin at each iteration. The inverse operator corresponding algorithms are based on the a priori information of the image to be reconstructed. The projection algorithm remains almost the same as classical Mojette transform algorithm.

3.1 Projection

The image composed of a few non-zero pixels is first projected using a set of (p,q) angles. On each projection, the bins values represents the number of non-zero pixels on the projection line.

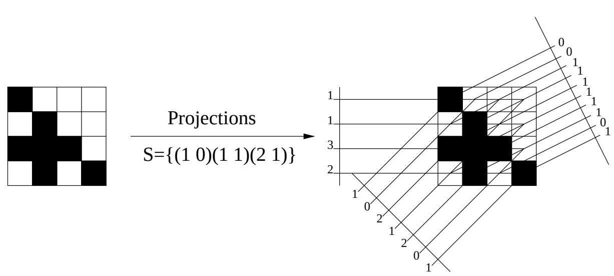

Figure 2: Exemple of projection of a binary shape

In the following example, a 64×64 Lena edges image shown in figure 3 has been projected according to the projection set : $S = \{(1\,0); (0\,1); (1\,1); (1\,-1); (20\,7); (15\,4)\}$. The number of zero value pixels is 2828 (for 4096 pixels). The total number of bin is 3282 where 1219 bins were zero valued.

Figure 3: Binary 64 × 64 image of Lena

3.2 Exact backprojection

3.2.1 *Regular algorithm*

The basic principles of the backprojection algorithm are the following : first n-to-one matches between pixels and bins have to be found, then the bins are backprojected and finally the projections set is updated.

If a bin is zero-valued, then all the pixels (carried by a discrete projection line) matching with this bin are also zero-valued. In the case where the bin value is $n \neq 0$, n is compared to the amount of pixels on the projection line not yet recontructed. If these two values match, then the bin can be backprojected.

```
for indexbin ← 1 to B
  if bin.binary = 0 or n
   {(n,b)} ← Step1
   {(lig,col)} ← Step2{(n,b)}
   binary(lig,col) ← bin(p,q,b)
   for indexprojection ← 1 to I
    update corresponding bin
   endfor
  endif
endfor
```

Step 1 List of n-to-one correspondences between bins and pixels belonging to the projection line and denoted by {(n,b)}.

Step 2 Linking information between bins and pixels allows to easily retrieve pixel coordinates

3.2.2 *Optimized algorithm*

This binary reconstruction can be split into two parts. First "obvious" zero-value pixels are reconstructed. If a bin has a zero value, this means that no pixel with a non-zero value was found on this projection line. Accordingly, all the pixels belonging to this projection line are set to zero. When projections are judiciously chosen, there is a high ratio of zero value bins on this projections. Indeed this first part is quickly computed as there is no updates on the projections to be done (backprojecting zeros does not modify bins values).

In the second part, the aim is to mark the pixels belonging to the shape with value one. This part is processed using the algorithm presented above.

3.3 Grey level reconstruction

When the previous binary reconstruction allows to reconstruct the binary shape, a second step aiming at retrieving the grey levels values belonging to the shape can start. This final step is actually a classical inverse Mojette operator. A one-to-one relationship between pixel and bin is first searched for, then the matching pixel gets the bin value, finally projections are updated. This process is iterated, until complete reconstruction is achieved.

Notice that each step is still of complexity order $O(IN)$, i.e. the overall complexity is $O(2IN + \epsilon)$ where ϵ is the cost of the first step (obvious zero values reconstruction). Each step of the global algorithm is subject to failure. This means that even if if a projection set managed to reconstruct the binary, the grey levels values may not be retrieved by the same set.

4 CHOICE OF THE PROJECTIONS SETS

When using the classical Mojette transform, Katz's criterion [4] has to be respected to ensure reconstruction. Let I be a $P \times Q$ image and $S = \{(p_i, q_i)\}$ a set of projection, the reconstruction is possible if $\sum_i |p_i| \geq P$ or $\sum_i |q_i| \geq Q$. Thanks to our a priori knowledge on the images or volumes, Katz's rule can be passed over for the discrete backprojection operator.

In this section we will discuss the choice of the projection set in the case of exact backprojection operator.

4.1 Conjecture

In this paragraph we will introduce a conjecture that allows to decide if a set of projection is sufficient to complete the reconstruction on the shape.

First, consider that S allows for the reconstruction of I. Now let S_1 be a sub set of S, $S_1 = S \setminus (p_j\ q_j)$ and G be the ghost [3] generated by dilatations of the 2PSE [1].

Conjecture

> If G or a linear combination of G can be mapped on the erosion of I by (p_j, q_j) without modifying the image values, then (p_j, q_j) is not necessary for reconstruction using the line backprojection operator.

Knowing that the image dynamic must be respected, coefficients for the linear combination of G can only be in $\{-1, 0, 1\}$. Moreover "-1" (resp. 1) value of the ghost must match with "0" (resp. 1) pixel value.

We will use the exemple of fig. 4. The binary shape to reconstruct is 4×4 large. Thanks to Katz's criterion, $S = \{(1\ 1)(1\ 0)(2\ 1)\}$ is ensured to reconstruct the shape. Let's try to determine if $S_1 = S \setminus (2\ 1)$ also allows the reconstruction.

The above figure represents two different shapes and the ghost that have to be mapped on the shapes.

In the first case (shape 1) the ghost cannot be mapped onto the image. Then angle (2 1) is not necessary for reconstruction and S_1 is sufficient.

[1] **2PSE** stands for Two Pixel Structuring Elements

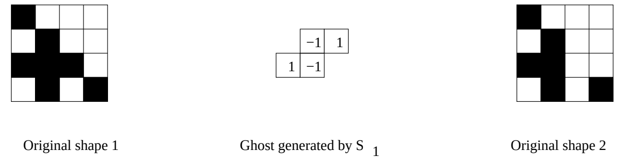

Figure 4: Two different shapes and a ghost

In the second case, the original image has been changed so that the ghost can be mapped on it. Here, the image cannot be reconstructed with S_1. Thus projection (2 1) is necessary for reconstruction.

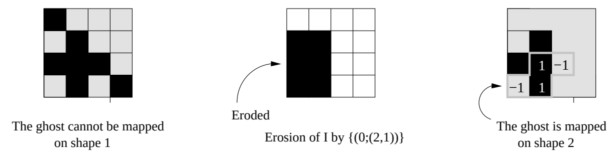

Figure 5: Mapped ghosts an the eroded shape

The two previous cases are quite simple. Nonetheless, the difficult practical point is to find the right linear combination for the ghost. For bigger shape, the number of different combination exponentially increases with the number of pixels of the ghost and image. Then, we next suggest another way to determine the projection set using an n-ary tree.

4.2 N-ary tree

The root of this tree will be a set fulfilling Katz's criterion. Let this set contains N different projections. Each level of the tree will be constituted of subsets of the root set. Then we recursively try to reconstruct the binary support using the different subsets generated by the n-ary. When the reconstruction failed, the sub-tree does not need to be pruned further. Finally we get all the smallest sub-sets allowing the reconstruction of the binary support. The best choice is then computed by choosing the set with the minimal bin number.

Next is an example of tree construction. Let I be a small 8×8 image. The base set of projection (the root node of the tree) is $S_0 = \{(4\ 1), (2\ 1), (1\ 1), (1\ 0)\}$. In this example, the root node allows the reconstruction of the image I. The first three sub-sets of the first level of the tree also allows for reconstruction, whereas the forth one ($\{(2\ 1); (1\ 1); (1\ 0)\}$) does not. Concerning the second level, none of the sub-sets allows for the recovery of the original support. Finally, the first three sub-sets of level 2 are the smallest sub-sets possible for reconstruction. The one with the smallest amount of bins should be preferred as it is also the smallest in terms of data storage ($\{(4\ 1)(1\ 1)(1\ 0)\}$).

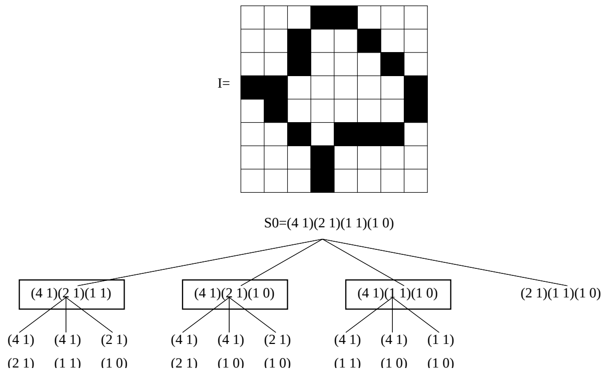

Figure 6: Example of an n-ary tree construction.

5 APPLICATION

The most natural application in image analysis for this new operator is the description and reconstruction of sparse images.

In figure 3.1 the 64×64 Lena image is a binarized edges images. In the following example the same image is valuated. Then the image is projected according to a set $S = \{(1\ 0); (0\ 1); (1\ 1); (1\ -1); (20\ 7); (15\ 4)\}$. According to Katz's criterion this set could not reconstruct the image using the classical Mojette transform. When using the new operator the set S allows to reconstruct the image. In the next figure, we show the original grey levels image, and the results the exact backprojection algorithm, figure 5.b) show the result for the first part of the optimized algorithm, we can notice that a lot of zero value pixel are quickly reconstructed after this first part. Figure 5.c) and d) show the reconstructed binary and grey levels image.

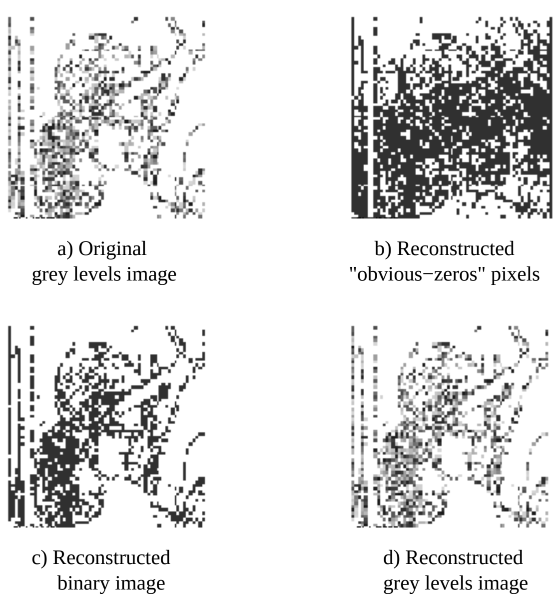

Figure 7: Example of backprojection with Lena image

This new version of Mojette transform can also be applied on binary image generated from bitstream. Let B be a bitstream containing 16384 (16Ko). This bitstream can be mapped onto a 128×128 binary image. Let consider that the a priori knowledge is a very unbalanced probabilities about the distribution of "1" and "0" pixels. In our example, it contains 10% "1" pixels and 90% "0" pixels.

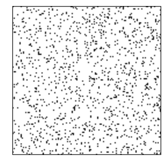

Figure 8: A bitstream mapped on a 128×128 buffer

This bitstream can be reconstructed with the set $S = \{(1\ 1); (1\ 16); (1\ 32); (1\ 12)\}$, which represents only 8256 bins, i.e. $red = -0.496$.

6 CONCLUSION

In this paper, a new way of representing binary images using a specific implementation of the Mojette transform was presented. The goal was to represent and exactly reconstruct 2D edges of an original binary or grey levels pixel set. An original algorithm of an exact backprojetor was presented, which is linear both in pixel and projection number. The projection is image-content-dependent and suppose high number of zero value pixels. An algorithm for choosing the minimum set of projection was also presented. Examples to real edges images and to bitstream mapped onto geometrical buffer were given. RLE encoding of the projections can be a second stage of compression for the proposed method.

References

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