BACKWARD ADAPTIVE TRANSFORM CODING OF VECTORIAL SIGNALS : A COMPARISON BETWEEN UNITARY AND CAUSAL APPROACHES

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ABSTRACT

In a transform coding framework we compare the optimal unitary approach (Karhunen-Loeve Transform, KLT) to the optimal causal approach (LDU, Lower-Diagonal-Upper). In absence of perturbations, both transforms have recently been shown to yield the same coding gain [2, 4]. The purpose of this paper is to compare the behavior of these two transformations when the ideal transform coding scheme gets perturbed, that is, when only a perturbed value $R_{XX} + \Delta R$ of R_{XX} is known at the encoder. In a real backward adaptive scheme, ΔR is due to two noise sources : estimation noise (finite set of available data at the encoder) and quantization noise (quantized data at the decoder). Furthermore, not only the transformation itself gets perturbed, but also the bit assignment. Theoretical expressions for the coding gains in both unitary and causal approaches are derived. Simulation results confirming the predicted behavior of the coding gains with perturbations are reported. This work is a follow-up of [1], where the influences of quantization and estimation noises were analyzed separately.

1 INTRODUCTION

Consider a stationary Gaussian vectorial source $\{X\}$. This source may be composed of any scalar sources $\{x_i\}$. In the classical transform coding framework, a linear transformation T is applied to each N-vector X_k to produce an N-vector $Y_k = TX_k$ whose components are independently quantized using scalar quantizers Q_i. A number of bits r_i is attributed to each Q_i under the constraint $\sum_i r_i = Nr$. For an entropy constrained scalar quantizer of a Gaussian source, the high resolution distortion is $\mathrm{E}(y^q_{i,k} - y_{i,k})^2 = \sigma^2_{q_i} = c2^{-2r_i}\sigma^2_{y_i}$, where $c = \frac{\pi e}{6}$.

An important property of commonly used transformations is that, if a noise (for example quantization noise) is added to the signal Y, then its power will be the same in the transform and in the signal domains. This property is sometimes referred to as "unity noise gain" property [2]. The coding gain for T is then defined as

$$G_T = \frac{E\|\tilde{X}\|^2_{(I)}}{E\|\tilde{X}\|^2_{(T)}} = \frac{E\|\tilde{X}\|^2_{(I)}}{E\|\tilde{Y}\|^2_{(T)}}, \qquad (1)$$

where I is the identity matrix, and the notation $\|\tilde{X}\|^2_{(T)}$ denotes the variance of the quantization error on the vector X, obtained for a transformation T. The optimal bit assignment yields the well known distortion for the vectorial signal $\{Y\}$: $\mathrm{E}\|\tilde{Y}\|^2_T = \frac{1}{N}\sum_{i=1}^N \sigma^2_{q_i} = Nc2^{-2r}\left(\prod_{i=1}^N \sigma^2_{y_i}\right)^{\frac{1}{N}} = N\sigma^2_q$. $\sigma^2_{q_i}$ is independent of i, and the number of bits assigned to the ith component is $r + \frac{1}{2}log_2\frac{\sigma^2_{y_i}}{(\prod_{i=1}^N \sigma^2_{y_i})^{\frac{1}{N}}}$. In the next section, we recall the main characteristics of the optimal causal approach (LDU) when optimized on R_{XX}, and recall why its performance is the same as the best unitary approach (KLT).

However, a backward adaptive coding scheme generally deals with non- or locally stationary signals. Sending the updates of the signal-dependent transformation and bit assignment as side information would cause a considerable overhead for the total bit rate. Thus, the backward adaptive coding scheme should require that neither the transformation nor the parameters of the bit assignment are transmitted to the decoder. So suppose now that the coding scheme is based on $\widehat{R}_{XX} = R_{XX} + \Delta R$ instead of R_{XX}, where $\widehat{R}_{XX}$ is available at both encoder and decoder. Then the computed transformation will be $\widehat{T} = T + \Delta T$, and the distortion will, under high resolution assumption, be proportional to the variances of the signals transformed by means of $\widehat{T}$ instead of T, say $\sigma^{2'}_{y_i}$. Moreover, the bits r_i should be attributed on the basis of estimates of the variances available at both encoder and decoder also, that is, $(\widehat{T}\widehat{R}_{XX}\widehat{T})_{ii}$, where $(.)_{ii}$ denotes the ith diagonal element of $(.)$. Hence, under the assumption of Gaussianity for the transformed signals , we get the following measure of distortion for

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a transformation $\widehat{T}$ based on $\widehat{R}_{XX}$:

$$E\|\tilde{Y}\|^2_{(\widehat{T})} = E\sum_{i=1}^{N} c2^{-2[r+\frac{1}{2}log_2 \frac{(\widehat{T\widehat{R}_{XX}\widehat{T}^T})_{ii}}{(\prod_{i=1}^{N}(\widehat{T\widehat{R}_{XX}\widehat{T}^T})_{ii})^{\frac{1}{N}}}]}\sigma^{2'}_{y_i}, \quad (2)$$

where the expectation is w.r.t ΔR in case it is non-deterministic. As a preliminary study [1], this distortion was computed for LDU and KLT transforms for a ΔR caused by quantization noise only (the coding scheme is based on the statistics of the quantized data, under high resolution assumption) and by estimation noise only (the coding scheme is based on an estimate of R_{XX} computed with a finite amount of K vectors). For the sake of clarity, the results concerning these two cases are briefly summarized in the third and fourth sections respectively. Section 3 also shows that the expression of the gain in the causal case with quantization noise in [1] is a lower bound, but that the actual coding gain however is very close to this bound. The main results concerning the joint influence of quantization and estimation noise for the coding gains of KLT and LDU are presented in the fifth part. Detailed calculations involved in this work will be skipped for lack of space but can be found (as those leading to the results of [1]) in [5]. Simulation results confirming these theoretical gains are presented in Section 6.

2 OPTIMAL CAUSAL AND UNITARY APPROACHES WITHOUT PERTURBATION

In the causal case, $Y = LX = X - \overline{L}X$, where $\overline{L}X$ is the reference vector. The output X^q is $Y^q + \overline{L}X$. As detailed in [4, 2], the components y_i are the prediction errors of x_i with respect to the past values of X, the $X_{1:i-1}$, and the optimal coefficients $-L_{i,1:i-1}$ are the optimal prediction coefficients. It follows that $R_{XX} = L^{-1}R_{YY}L^{-T}$, which represents the LDU factorization of R_{XX}. The coding gain is for L is

$$G_L^{(0)} = \left(\frac{\det[diag(R_{XX})]}{\det[diag(LR_{XX}L^T)]}\right)^{\frac{1}{N}} = \left(\frac{\det[diag(R_{XX})]}{\det\Lambda}\right)^{\frac{1}{N}} = G_V^{(0)}, \quad (3)$$

where V denotes a KLT of R_{XX}, and Λ its corresponding eigenvalue matrix. This is actually the best coding gain achievable among all unimodular transforms (a proof can be found in [3]).

3 QUANTIZATION EFFECTS ON THE CODING GAINS

Suppose we compute the transformation on the basis of quantized data (whose statistics are assumed to be perfectly known in this section). Under the assumptions of high resolution (uncorrelated white noise), optimal bit assignment and unity noise gain property of the transformation, ΔR equals $E\tilde{X}\tilde{X}^T = \sigma_q^2 I$, where $\sigma_q^2 = c2^{-2r}\left(\prod_{i=1}^{N}\sigma_{y_i}^2\right)^{\frac{1}{N}}$ for Gaussian transform signals. Thus,

$$E\|\tilde{Y}\|^2_{(\widehat{T},q)} = \sum_{i=1}^{N} c2^{-2[r+\frac{1}{2}log_2 \frac{(\widehat{T}R_{X^qX^q}\widehat{T}^T)_{ii}}{(\prod_{i=1}^{N}(\widehat{T}R_{X^qX^q}\widehat{T}^T)_{ii})^{\frac{1}{N}}}]}\sigma^{2'}_{y_i}. \quad (4)$$

(where q refers to quantization). Evaluating (4) for $\widehat{T} = I, \widehat{V}$ and $\widehat{L}$ gives the following results [1].

3.1 KLT

For $\widehat{T} = I, \widehat{V}$, the quantization does not change transformations. However, an increase in distortion comes from the perturbation occuring on the bit assignment mechanism. One shows that the coding gain based on quantized data is in the unitary case

$$G_{\widehat{V},q} = G^0 \frac{(\det(I + \sigma_q^2(diagR_{XX})^{-1}))^{\frac{1}{N}} tr\{\left(I + \sigma_q^2(diagR_{XX})^{-1}\right)^{-1}\}}{(\det(I + \sigma_q^2(\Lambda^{-1})))^{\frac{1}{N}} tr\{(I + \sigma_q^2(\Lambda^{-1}))^{-1}\}}. \quad (5)$$

3.2 LDU

In the causal case, the coder uses a transformation $\widehat{L} = L'$ such that $L'R_{X^qX^q}L'^T = R'_{YY}$. R'_{YY} is the diagonal matrix of the estimated variances involved in the bit assignment (L' and R'_{YY} are both available to the decoder). In this case, the difference vector Y is $X - \overline{L'}X^q$: the quantization noise is filtered by the rows of $\overline{L'}$, see Figure 2. Note that $E\|\tilde{X}\|^2_{L',q}$ still equals $E\|\tilde{Y}\|^2_{L',q}$,

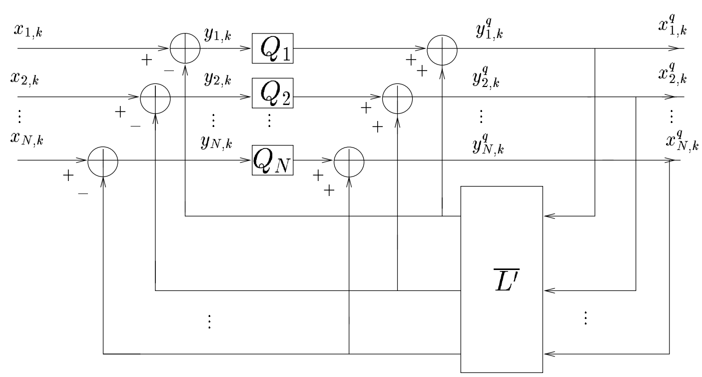

Figure 1: Closed Loop Causal Coding Scheme.

since $\tilde{X} = X^q - X = Y^q + \overline{L'}X^q - X = Y^q - (X - \overline{L'}X^q) = Y^q - Y = \tilde{Y}$. The actual variance of the ith signal y_i obtained with L' is $\sigma^2_{y'_i} = (L'R_{X^qX^q}L'^T - \sigma_q^2 I)_{ii}$ [4]. We now show that evaluating (4) with $\widehat{T} = L'$ yields an upper bound for $E\|\tilde{Y}\|^2_{(L',q)}$, and thus a lower bound for the coding gain $G_{L',q}$ in the causal case. Consider the signal in the transform domain :

$$y_{i,k} = x_{i,k} - \sum_{j=1}^{i-1}\overline{L'}_{ij}x_{i-j,k} - \sum_{j=1}^{i-1}\overline{L'}_{ij}q_{i-j,k}. \quad (6)$$

The random variable (r.v.) y_i is the sum of a Gaussian r.v. $z_{i,k} = x_{i,k} - \sum_{j=1}^{i-1} \overline{L}'_{ij} x_{i-j,k}$, and of $i-1$ uniform r.v.s $u_{ij,k} = \overline{L}'_{ij} q_{i-j,k}$. Hence, for $i > 1$, $\{y_i\}$ is not Gaussian. In order to compare the actual rate-distortion function of the $\{y_i\}$ with that of a Gaussian r.v., denote by $H(y_i^q)$ the entropy of the quantized variable y_i^q, $h(y_i)$ the differential entropy of y_i, r_i the minimum number of bits per sample necessary to code losslessly y_i^q, and Δ_i the step size of the uniform quantizer Q_i. Then we have under the high resolution assumption $H(y_i^q) = r_i \approx h(y_i) - log_2\Delta_i$, where $log_2\Delta_i$ corresponds to the differential entropy of the uniformly distributed r.v. q_i : $log_2\Delta_i = h(q_i) = \frac{1}{2}log_2(12\sigma_{q_i}^2) = \frac{1}{2}log_2(12E\|\tilde{y}_i\|^2)$. Thus $r_i \approx h(y_i) - \frac{1}{2}log_2(12E\|\tilde{y}_i\|^2)$. Since for a given variance the Gaussian probability density function (p.d.f) maximizes the differential entropy $h(y_i)$, an upper bound for r_i may be found as $r_i < \frac{1}{2}log_2\frac{2\pi e\sigma_{y_i'}^2}{12E\|\tilde{y}_i\|^2}$, which gives $E\|\tilde{y}_i\|^2 < \frac{\pi e}{6}2^{-2r_i}\sigma_{y_i'}^2$. The distortion $E\|\tilde{Y}\|_{(L',q)}^2$ is then upper bounded by (4), that is, with $c = \frac{\pi e}{6}$,

$$E\|\tilde{Y}\|_{(L',q)}^2 = \sum_{i=1}^N E\|\tilde{y}_i\|^2$$
$$< \sum_{i=1}^N c2^{-2r_i}\sigma_{y_i'}^2 = \sum_{i=1}^N c2^{-2[r+\frac{1}{2}log_2\frac{(R'_{YY})_{ii}}{(\prod_{i=1}^N (R'_{YY})_{ii})^{\frac{1}{N}}}]}\sigma_{y_i}^{2'}. \quad (7)$$

Hence, the gain $G_{L',q} = \frac{E\|\tilde{Y}\|_{(I,q)}^2}{E\|\tilde{Y}\|_{(L',q)}^2}$ becomes lower bounded by $\frac{E\|\tilde{Y}\|_{(I,q)}^2}{\sum_{i=1}^N c2^{-2r_i}\sigma_{y_i'}^2} = G_{L',q}^{low}$. It is shown in [1] that $G_{L',q}^{low}$

$$= G^0\frac{(\det(I+\sigma_q^2(diagR_{XX})^{-1}))^{\frac{1}{N}}tr\{(I+\sigma_q^2(diagR_{XX})^{-1})^{-1}\}}{(\det(I+\sigma_q^2(\Lambda^{-1})))^{\frac{1}{N}}tr\{(I-\sigma_q^2(R_{YY'}^{-1}))\}}. \quad (8)$$

Now, the r.v. y_1 is strictly Gaussian. Moreover, the convolution of $i-1$ Uniform p.d.f.s is known to tend quickly, as i grows, to a Gaussian p.d.f. Thus, y_i tends to be Gaussian and for reasonably high N, this bound is a fairly precise measure of the actual coding gain : $G_{L',q} \approx G_{L',q}^{low}$.

4 ESTIMATION NOISE

We present in this section the coding gains of a backward adaptive scheme based on an estimate of the covariance matrix $\widehat{R}_{XX} = \frac{1}{K}\sum_{i=1}^K X_iX_i^T$. We suppose independent identically distributed real vectors X_i, which is for example the case if the sampling period of the scalar signals is high in comparison with their typical correlation time. In the causal case, there is a qualitative difference with the previous section, where the quantization noise was filtered by the predictors of $\overline{L'}$. Here, the estimation noise does not perturb signals (in particular the y_i are Gaussian even in the causal case), but only transformations and bit assignments. An interesting result (due to the decorrelation and unimodularity properties of the two transformations) is that the coding gains with estimation noise are the same in the causal and the unitary cases. With $D = diag\{R_{XX}\}$, the coding gain for the two approaches, by computing the transforms and the bit assignments on the basis of K vector is given by

$$G_{\widehat{V},K} = \frac{E\|\tilde{Y}\|_{(I,K)}^2}{E\|\tilde{Y}\|_{(\widehat{V},K)}^2} \approx G^0\left(1 - \frac{1}{K}\left[\frac{tr\{RD^{-1}RD^{-1}\}}{N^2} + \frac{N-1}{2} - \frac{1}{N}\right]\right). \quad (9)$$

5 QUANTIZATION AND ESTIMATION NOISE

We arrive now at the most general case of this comparison between causal and unitary approaches. As stated in the Introduction, in a real backward adaptive scheme, the coder should attribute the bits on the basis of $\widehat{R}_{X^qX^q} = R_{X^qX^q} + \Delta R = \frac{1}{K}\sum_{i=1}^K X_i^q X_i^{qT}$. As in the previous section, we assume independent identically distributed real vectors X_i. The estimated transform is $\widehat{T}$, such that $\widehat{T}\widehat{R}_{X^qX^q}\widehat{T}^T$ is a diagonal matrix, which corresponds to the estimated variances of the transformed signals. If we continue denoting by $\sigma_{y_i'}^2$ the actual variances of the transformed signals (obtained by applying $\widehat{T}$ to X), the expected distortion obtained with $\widehat{T}$ using K quantized vectors is

$$E\|\tilde{Y}\|_{(\widehat{T},K,q)}^2 = E\sum_{i=1}^N c2^{-2[r+\frac{1}{2}log_2\frac{(\widehat{T}\widehat{R}_{X^qX^q}\widehat{T}^T)_{ii}}{(\prod_{i=1}^N(\widehat{T}\widehat{R}_{X^qX^q}\widehat{T}^T)_{ii})^{\frac{1}{N}}}]}(\widehat{T}R_{XX}\widehat{T}^T)_{ii}, \quad (10)$$

where the subscripts q and K refer to the presence of quantization and estimation noise, and the constant c assumes Gaussianity of the transformed signals. Equation (10) must be evaluated for the Identity, KLT and LDU transforms. As in Section 3, the computation of (10) in the causal case will provide an upper bound for the distortion because of the uniform quantization noise feedback upon the y_i. The details of the calculations will be skipped in the following for lack of space but can be found in [5]. Simulation results of these theoretical results are presented in Section 6.

5.1 Identity Transformation

In this case the transformed signals y_i are indeed still Gaussian. Under the high resolution assumption, the expected distortion (10) for Identity with quantization and estimation noise is, for sufficiently high K

$$E\|\tilde{Y}\|_{(I,K,q)}^2 \approx E\|\tilde{Y}\|_{(I)}^2\left(\det(I+\sigma_q^2(diag\{R_{XX}\})^{-1})\right)^{1/N}$$
$$\left[1+\frac{1}{K}\left[1-\frac{tr\{R_{X^qX^q}D^{q-1}R_{X^qX^q}D^{q-1}\}}{N^2}\right]-\frac{\sigma_q^2}{N}tr\{(diagR_{X^qX^q})^{-1}\}\right]. \quad (11)$$

5.2 KLT

In this case also, expression (10) with $\widehat{T} = \widehat{V}$ gives the exact expression of the distortion since each transform coefficient is a linear combination of Gaussian r.v.s..

Evaluated in the unitary case, this distortion with quantization and estimation noise may be approximated as

$$E\|\tilde{Y}\|^2_{(\widehat{V},K,q)} \approx E\|\tilde{Y}\|^2_{(K)} \left(\det(I+\sigma_q^2(R_{XX})^{-1})\right)^{1/N} \times \left[1+\frac{1}{K}\left[\frac{N-1}{2}+\frac{N-1}{N}\right] - \frac{\sigma_q^2}{N}tr\{(\Lambda^q)^{-1}\}\right], \qquad (12)$$

where $\Lambda^q = VR_{X^qX^q}V^T = \Lambda + \sigma_q^2 I$. With $D^q = diag\{R_{X^qX^q}\}$, one can show that the corresponding expression for the coding gain is

$$G_{(\widehat{V},K,q)} = \frac{E\|\tilde{Y}\|^2_{(\widehat{V},K,q)}}{E\|\tilde{Y}\|^2_{(\widehat{V},K,q)}} \approx G^0 \frac{\left(\det(I+\sigma_q^2(diag\{R_{XX}\})^{-1})\right)^{1/N}}{\left(\det(I+\sigma_q^2(R_{XX})^{-1})\right)^{1/N}} \times \frac{\left[1+\frac{1}{K}\left(1-\frac{tr\{R_{X^qX^q}D^{q-1}R_{X^qX^q}D^{q-1}\}}{N^2}\right)-\frac{\sigma_q^2}{N}tr\{(diagR_{X^qX^q})^{-1}\}\right]}{\left[1+\frac{1}{K}\left(\frac{N-1}{2}+\frac{N-1}{N}\right)-\frac{\sigma_q^2}{N}tr\{(\Lambda^q)^{-1}\}\right]}. \qquad (13)$$

5.3 LDU

In the causal case, an estimate $\widehat{L}'$ of L is computed, and each r.v. is not purely Gaussian :

$$y_{i,k} = x_{i,k} - \sum_{j=1}^{i-1} \widehat{L'}_{ij} x_{i-j,k} - \sum_{j=1}^{i-1} \widehat{L'}_{ij} q_{i-j,k}. \qquad (14)$$

Similarly as in (7), the expected distortion for the LDU $E\|\tilde{Y}\|^2_{(\widehat{L}',K,q)}$ is upper bounded by the rate-distortion function of a set of Gaussian r.v.s of same variances $\sigma^2_{y'_i}$, that is

$$E\|\tilde{Y}\|^2_{(\widehat{L}',K,q)} < \sum_{i=1}^{N} c 2^{-2\left[r+\frac{1}{2}log_2\frac{(\widehat{L}'\widehat{R}_{X^qX^q}\widehat{L}'^T)_{ii}}{(\prod_{i=1}^{N}(\widehat{L}'\widehat{R}_{X^qX^q}\widehat{L}'^T)_{ii})^{\frac{1}{N}}}\right]} \sigma^{2'}_{y_i}, \qquad (15)$$

with $\sigma^{2'}_{y_i} = (\widehat{L}'R_{X^qX^q}\widehat{L}'^T - \sigma_q^2 I)_{ii}$. Thus, computing (10) when the transformation is based on K quantized vectors (for high K and under high resolution assumption) gives an upper bound, which is

$$E\|\tilde{Y}\|^2_{(\widehat{L}',K,q)} < E\|\tilde{Y}\|^2_{(L)} \left(\det(I+\sigma_q^2(R_{XX})^{-1})\right)^{1/N} \times \left[1+\frac{1}{K}\left[\frac{N-1}{2}+\frac{N-1}{N}\right] - \frac{\sigma_q^2}{N}tr\{(R'_{YY})^{-1}\}\right], \qquad (16)$$

where $R'_{YY} = L'R_{X^qX^q}L'^T$. The corresponding expression for the coding gain in the causal case is then lower bounded by

$$G_{(\widehat{L}',K,q)} = \frac{E\|\tilde{X}\|^2_{(I,K,q)}}{E\|\tilde{Y}\|^2_{(\widehat{L}',K,q)}} > G^0 \frac{\left(\det(I+\sigma_q^2(diag\{R_{XX}\})^{-1})\right)^{1/N}}{\left(\det(I+\sigma_q^2(R_{XX})^{-1})\right)^{1/N}} \times \frac{\left[1+\frac{1}{K}\left[1-\frac{tr\{R_{X^qX^q}D^{q-1}R_{X^qX^q}D^{q-1}\}}{N^2}\right]-\frac{\sigma_q^2}{N}tr\{(diagR_{X^qX^q})^{-1}\}\right]}{\left[1+\frac{1}{K}\left[\frac{N-1}{2}+\frac{N-1}{N}\right]-\frac{\sigma_q^2}{N}tr\{(R'_{YY})^{-1}\}\right]}. \qquad (17)$$

As in Section 3, since the r.v. y_i tends to be Gaussian very quickly as i grows, the bound in (17) is a fairly precise measure of the actual coding gain $G_{(\widehat{L}',K,q)}$ for reasonably high N.

Indeed, it can be checked that the expression (17) and (13) tend to (5) and (8) respectively as $K \to \infty$, and both tend to (9) as $\sigma_q^2 \to 0$.

6 SIMULATIONS

For the simulations, we generated real Gaussian i.i.d. vectors with covariance matrix $R_{XX} = HR_{AR1}H^T$. R_{AR1} is the covariance matrix of a first order autoregressive process with normalized correlation coefficient ρ. H is a diagonal matrix whose ith entry is $(N-i+1)^{1/3}$. We assumed entropy constrained scalar quantizers Q_i with high resolution rate-distortion function $\sigma^2_{q_i} = \frac{\pi e}{6}2^{-2r_i}\sigma^2_{y_i}$. Thus, the Gaussianity of the transform signals was assumed in all cases. The coding gains in presence of estimation noise and quantization are compared for LDU and KLT in Figure 2, for $N = 8$, $\rho = 0.9$ and a rate of 3 bits per sample (mean over 100 realizations). The theoretical gains are given by (13) and (17). The observed behavior of the transformations corresponds quite well to the theoretically predicted one for $K \approx$ a few tens.

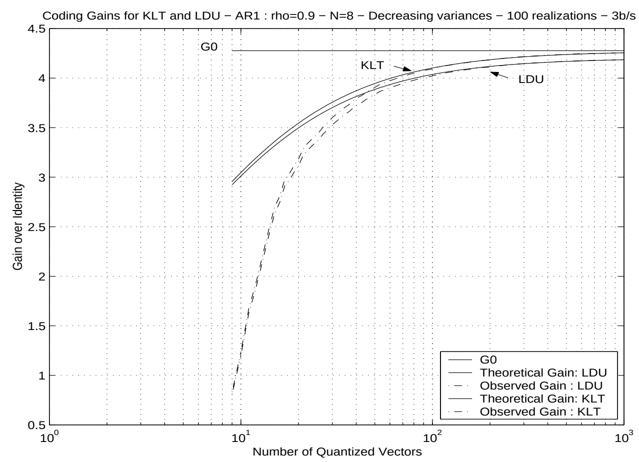

Figure 2: Gains for KLT and LDU vs K.

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