Structure Sensitive Sampling of Noisy Fractal Patterns

Kohji Kamejima
Osaka Institute of Technology
Osaka, Japan

ABSTRACT

An adaptive sampling scheme is presented for detecting complex patterns in noisy imagery. By representing imaging processes in terms of unknown contraction mappings, capturing probability is generated for fractal attractors approximating observed patterns. The totality of local maxima of the capturing probability is shown to yield a pattern sensitive sampling of the fractal attractor. The detectability of the sampling scheme has been verified through simulation studies.

1 Introductory Remarks

In the theory of digital image analysis, complex patterns are described as computable entities on discrete image plane. In many practical applications, such discrete representation should maintain complete information for exact restoration of complex imagery as shown in Fig. 1. In this figure, the expansion of roadway area is represented by random distribution of feature points of texture patterns. Within conventional statistical – computational frameworks, however, it is not easy to generate such "visible" representation of random imagery. For instance, sampling on "very fine" lattice often yields "fragile" discrete representation that is too susceptive to non-essential pattern deformation. By coarse sampling, on the other hand, it is not easy to identify observed pattern in near misses. To extract complex patterns within noisy background, thus, feature points should be sampled for regenerating exact patterns to be detected.

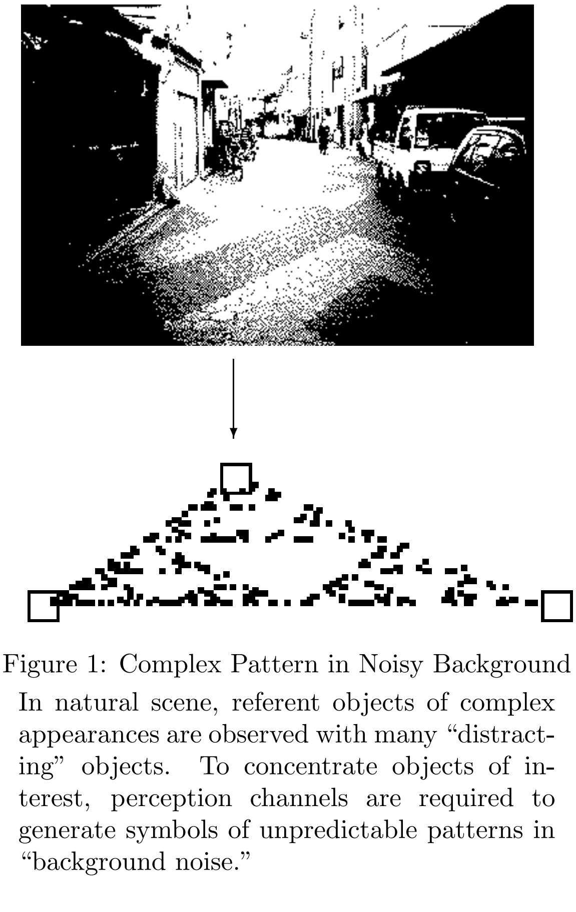

Figure 1: Complex Pattern in Noisy Background

In natural scene, referent objects of complex appearances are observed with many "distracting" objects. To concentrate objects of interest, perception channels are required to generate symbols of unpredictable patterns in "background noise."

A potential way to bypass the self-contradiction is to introduce the self-similarity as *a priori* pattern structure. Noticing logical – geometric coordination in self-similarity imaging processes, in this paper, we assume that patterns to be observed are generated as fractal attractors associated with unknown set of contraction mappings. The assumption of self-similarity is not restrictive because we can approximate any patterns via the following "Fractal Collage [1]": *For arbitrary pattern Λ in a fixed image plane Ω, there exists a set of contraction mappings $\nu = \{\mu_i, i = 1, 2, \ldots, m\}$, that yields an invariant subset $\Xi \subset \Omega$ for approximating the pattern Λ within arbitrary small imaging error.* This implies that any observed patterns can be coded in terms of finite symbols. The finite code completely specifies imaging process for generating fractal attractor of infinite geometric complexity.

In contrast with conventional statistical – computational representation, fractal model conveys complete information to specifies invoked contraction mappings. In fact, enough data for determining mapping parameter is visualized as the distribution of attractor points. Hence, we have logical bases for pattern coding as the following "Structural Observability [2]": *The attractor Ξ is cov-*

ered by the totality of fixed points θ_i, $\theta_j, \ldots$, associated with all finite composite of the mappings $\mu_1, \mu_2, \ldots, \mu_m$. Thus, pattern coding results in identifying origin – destination pairs in complex attractors. Since each attractor point deterministically "jumps" into the attractor by a contraction mapping, we have exact origin – destination associations in observed imagery.

In addition, the self-similarity induces definite association between geometric order, i.e., spatial distribution of attractor points, and probability for pattern capturing, i.e., brightness distribution. This implies that we can analyze pattern structure via the estimation of the "Invariant Measure [1]": *For arbitrary attractor Ξ generated by random application of fixed contraction mappings, there exists a measure $\chi_\Xi^\mathfrak{p}$ that is invariant with respect to transform by the mappings ν.* The existence of invariant measure implies the association between the distribution pattern and the density function of fractal attractors. The self-similarity of the density function introduces the self-similarity in the distribution of statistical parameter.

2 2D Gaussian Sampling

Let $\mathfrak{L}$ be a "uniform" lattice with resolution $\epsilon/\sqrt{2}$

$$\mathfrak{L} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & (i^-, j^+) & (i, j^+) & (i^+, j^+) & \cdots \\ \cdots & (i^-, j) & (i, j) & (i^+, j) & \cdots \\ \cdots & (i^-, j^-) & (i, j^-) & (i^+, j^-) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

$$i^\pm = i \pm \frac{\epsilon}{\sqrt{2}}, \qquad j^\pm = j \pm \frac{\epsilon}{\sqrt{2}},$$

and consider the integration of above mentioned three aspects of self-similarity on digital image plane. As discrete detectors of complex patterns, let the family of Gaussian probability density functions $\{g_\sigma^\ell, \sigma > 0\}$, be introduced on $\mathfrak{L}$ where

$$g_\sigma^\ell(\omega) = \frac{1}{2\pi\sigma} \exp\left[-\frac{|\omega - \ell|^2}{2\sigma}\right], \qquad \ell \in \mathfrak{L}. \quad (1)$$

Noticing g_σ^ℓ, $\sigma > 0$ yields a δ-convergent sequence:

$$g_\sigma^\ell \to \delta_\ell \quad \text{as} \quad \sigma \to 0,$$

where δ_ℓ denotes the "point" image concentrated on ℓ, we have the following stochastic sampling scheme on $\mathfrak{L}$:

$$\mathfrak{G} = \{g_\sigma^\ell, \ell \in \mathfrak{L}\}. \quad (2)$$

By testing the value of distributions on $\mathfrak{G}$, the image of the invariant measure $\chi_\Xi^\mathfrak{p}$ is exactly sampled in a stochastic sense:

$$\chi_\Xi^\mathfrak{G} = \{\chi_\Xi^\mathfrak{p}(g_\sigma^\ell)_\omega, \ell \in \mathfrak{L}\}, \quad (3a)$$

where

$$\chi_\Xi^\mathfrak{p}(g_\sigma^\ell)_\omega = g_\sigma^\ell * \chi_\Xi^\mathfrak{p}(\omega). \quad (3b)$$

In the sampling scheme $(\mathfrak{G}, \mathfrak{L})$, complete information $\chi_\Xi^\mathfrak{p}$ of infinite resolution is associated with discrete image plane $\mathfrak{L}$.

3 Associated Multi-Scale Image

Consider the adaptation of "scale parameter" σ to the self-similar pattern satisfying

$$\Xi = \bigcup_{\mu_i \in \nu} \mu_i(\Xi). \quad (4)$$

The structural observability implies that the process should be modeled by 2D dynamical system with the following antagonistic imaging mechanisms

- diffusion of point image δ_ξ within image plane Ω, and,
- successive reduction of imaging domain via not-yet-identified contraction mappings $\mu_i \in \nu$.

Let the imaging process model be described in terms of the following system

$$\exists \mu_i \in \nu \quad : \quad \omega_{t+1} = \mu_i(\omega_t), \quad (5)$$

where random shift of a point image is considered to be observed as a sample path on a "tectonic plate" successively reduced by randomly selected mapping $\mu_i \in \nu$. By identifying "observation error" with 2D Brownian motion, we have the following stochastic evaluation for capturing the point image within unknown attractor Ξ:

Proposition 1 (Capturing Probability) *Let $\chi_\Xi^\mathfrak{p}$ be a given brightness distribution to be collaged by $\nu = \{\mu_i\}$. Assume that the distribution is observed through the Gaussian array $\mathfrak{G}$. Then, for each $\ell \in \mathfrak{L}$, the probability for regenerating Ξ within the framework of maximum entropy capturing is visualized as a smooth field $\varphi(\omega|\nu)$ satisfying*

$$\frac{1}{2}\Delta\varphi(\omega|\nu) + \rho[\chi_\Xi^\mathfrak{p}(g_\epsilon^\ell)_\omega - \varphi(\omega|\nu)] = 0, \quad (6a)$$

where ρ is complexity factor specified in terms of $\|\nu\|$, the size of the set ν:

$$\rho = \log \|\nu\|. \quad (6b)$$

4 Pattern Boundary on Invariant Measures

By paraphrasing the generator (6) as

$$\varphi(g_\epsilon^\ell|\nu) = \chi_\Xi^\mathfrak{p}(g_\epsilon^\ell) + \frac{1}{2}\Delta\varphi(g_\epsilon^\ell|\nu) \cdot \tau(\rho), \quad (7)$$

where $\tau(\rho) = 1/\rho = 1/\log\|\nu\|$, we have the following association with multiscale imagery [3]:

$$\varphi((\cdot)|\nu) \quad \sim \quad g_{\tau(\rho)} \Longleftrightarrow g_\epsilon^\ell + \int_0^{\tau(\rho)} \frac{1}{2}\Delta g_t dt, \qquad (8)$$

on Gaussian array $(\mathfrak{G}, \mathfrak{L})$. Hence, we have the estimate $\hat{\Xi}_\nu$ on continuous image plane by

$$\hat{\Xi}_\nu \quad = \quad \Big\{\omega \in \Omega \mid \varphi(\omega|\nu) \geq \gamma(\rho)\Big\}. \qquad (9)$$

Consider self-similar patterns generated in noisy background. For such patterns, we can generate the capturing probability and a version of conditional probability for evaluating possible variation of brightness, as follows:

$$p(\omega|\nu) \quad = \quad \frac{\varphi(\omega|\nu)}{C_\Omega^\varphi}, \qquad (10a)$$

$$C_\Omega^\varphi \quad = \quad \int_\Omega \varphi(\omega|\nu)d\omega. \qquad (10b)$$

By using the conditional probability, we can index the complexity of brightness variation in terms of the following Shannon's entropy

$$\begin{aligned} \hat{H}_\nu &= -\int_\Omega p(\omega|\nu)\log p(\omega|\nu)d\omega \\ &= -\mathcal{E}\Big\{\log p(\omega|\nu) \mid \nu\Big\}. \qquad (11) \end{aligned}$$

The existence of self-similarity structure should be verified through the comparison with the entropy evaluation under "null condition":

$$\hat{H}_\emptyset \quad = \quad -\mathcal{E}\Big\{\log p(\omega|\emptyset) \mid \emptyset\Big\}, \qquad (12)$$

where $p(\omega|\emptyset) = \text{const.}$ on Ω. Hence, we have

Proposition 2 (Pointwise Filtering) [4] *Assume the background noise χ_Ω is uniformly distributed in the image plane Ω and suppose that observed measure χ_Λ is represented by*

$$\chi_\Lambda \quad = \quad \chi_\Xi^{\mathfrak{p}} + \chi_\Omega. \qquad (13)$$

Then the boundary level is given by

$$\gamma \quad = \quad C_\Omega^\varphi \bar{p}_\nu, \qquad (14a)$$

$$\log \bar{p}_\nu \quad = \quad 1 - \frac{1}{2}(1 - e^{\hat{H}_\nu - \hat{H}_\emptyset}) - \hat{H}_\emptyset. \qquad (14b)$$

5 Pattern Sensitive Sampling

The capturing probability $\varphi(\omega|\nu)$ is the smoothing of gray level distribution of self-similar patterns. By modeling the imaging via unknown contraction mappings in terms of 2D Brownian motion on dynamically regenerated domain, a unified framework is introduced for information compression: the maximum entropy. Due to the infinite differentiability of generated field, on the other hand, the capturing probability maintains complete information of self-similarity processes. The association (8), particularly, implies that the generator of the capturing probability is adapted to the complexity of the patterns to be observed.

Consider a discrete image defined by

$$\begin{aligned} \tilde{\Theta} \quad = \quad \Big\{\tilde{\theta} \in \Omega \Big| \nabla\varphi(\tilde{\theta}|\nu) = 0,& \\ \det\left[\nabla\nabla^T\varphi\right](\tilde{\theta}|\nu) > 0,& \\ \Delta\varphi(\tilde{\theta}|\nu) < 0\Big\}.& \qquad (15) \end{aligned}$$

Through the adaptation of the field $\varphi(\omega|\nu)$ to unknown generator, the image $\tilde{\Theta}$ yields a version of sampling pattern. On the sampled pattern $\tilde{\Theta}$, we can successively apply contraction mappings to generate complex pattern consisting of point images

$$\langle\mu_i\rangle_t(\tilde{\Theta}) \quad = \quad \Big\{\langle\mu_i\rangle_t(\tilde{\theta}) \mid \tilde{\theta} \in \tilde{\Theta}\Big\},$$

where $\langle\mu_i\rangle_t$ denotes a finite composite on mapping set ν with length t:

$$\langle\mu_i\rangle_t \quad = \quad \mu_i^t\mu_i^{t-1}\cdots\mu_i^2\mu_i^1, \qquad \mu_i^s \in \nu.$$

The mapping structure is said to be uniformly observable if, for arbitrary $\xi \in \Xi$ and $\epsilon > 0$, there exists a finite composite $\langle\mu_i\rangle_t$ generating the fixed point ξ_t

$$\xi_t \quad = \quad \langle\mu_i\rangle_t(\xi_t), \qquad \mu_i \in \nu, \qquad (16a)$$

such that the following condition is satisfied [5]:

$$|\xi - \xi_t| \quad < \quad \epsilon. \qquad (16b)$$

The observability condition can be tested on discrete image $\tilde{\Theta}$ as follows:

Proposition 3 (Invariant Features) *Assume that there exists an subset $\Theta \subset \tilde{\Theta}$ invariant with respect to ν, i.e.,*

$$\Theta \quad = \quad \Big\{\theta \in \tilde{\Theta} \mid \exists\mu_i \in \nu : \mu_i^{-1}(\theta) \in \Theta\Big\}. \qquad (17)$$

Suppose that for arbitrary $\mu_i \in \nu$ there exist $\theta^o, \theta^d \in \Theta$ such that

$$\theta^d \quad = \quad \mu_i(\theta^o). \qquad (18)$$

Then the image generator $\nu = \{\mu_i\}$ is uniformly observable.

Obviously, $\Theta \subset \Xi$ if $\tilde{\Theta} \subset \Xi$. This implies that we can restrict the domain for extracting invariant features by the following pointwise filter:

$$\hat{\Theta} \quad = \quad \Big\{\hat{\theta} \in \tilde{\Theta} \mid p(\hat{\theta}|\nu) \geq \bar{p}_\nu\Big\}. \qquad (19)$$

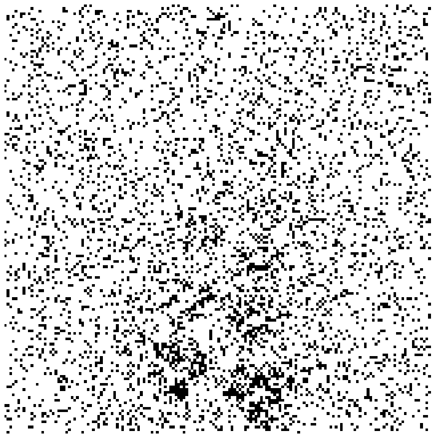

Figure 2: Noisy Observation of Fractal Leaves

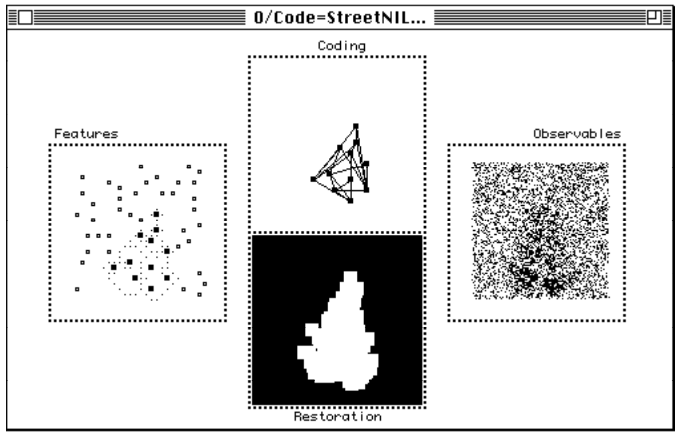

Figure 3: Uniformly Observable Fractal Model

Thus, we have generic representation of the self-similarity on noisy discrete imagery as follows:

$$\Theta \;=\; \Big\{\theta \in \hat{\Theta} \;\Big|\; \exists \mu_i \in \nu : \mu_i^{-1}(\theta) \in \Theta\Big\}. \qquad (20)$$

In this representation, the constraint for imaging process is grammatically specified on discrete pattern $\hat{\Theta}$. The discrete pattern $\hat{\Theta}$ is extracted within sampled image $\tilde{\Theta}$ through pointwise filtering. The discrete information $\tilde{\Theta}$, conversely, is generated through adaptive sampling based on stochastic evaluation $\varphi(\omega|\nu)$ for unknown mappings ν.

6 Experiments

Pattern detection on proposed sampling scheme was verified via simulation studies. In these simulations, fractal attractors were generated by Monte-Carlo simulation on continuous image model. To each attractors, uniformly distributed random dots were added as background noise as shown in Fig. 2. Results of simulation studies are illustrated in Fig. 3.

Figure 2 illustrate an observation of a fractal pattern $\chi_\Xi^{\mathfrak{p}}$ in background noise χ_Ω satisfying $\|\chi_\Omega\| = 4\|\chi_\Xi^{\mathfrak{p}}\|$. Pattern detection results in this situation are shown in Fig. 3. In these figures, the distribution of attractor points are "counted" on 2D lattice $\mathfrak{L}$ of 32×32 size (Observables view) where we have the initial value for generating capturing probability $\varphi(\omega|\nu)$. Extracted stochastic features are illustrated in "Features View" where $\hat{\Theta}$ is estimated via in-out discriminator (19) and indicated by (■) in background noise (□). As shown in Fig. 3, the generator of observed self-similar pattern is observable so that the generator yields invariant subset $\Theta \subset \hat{\Theta}$ (Coding View) and regenerates fractal attractor (Restoration View). Thus, we can detect the generator of observed pattern via structure sensitive sampling $\hat{\Theta}$ on discrete image $(\mathfrak{G}, \mathfrak{L})$.

Proposed detection method was applied to various fractal patterns in various levels of additive noise. The results of simulation studies are summarized as follows:

- Proposed pointwise filter generates discrete subset of unknown fractal attractors.
- Sampled patterns is well structured to support origin – destination associations with respect to not-yet-identified mapping set.
- Structural consistency of sampled pattern with mapping descriptions can be evaluated by observability test.

7 Concluding Remarks

A method was presented for structure sensitive sampling of unknown fractal attractors in noisy imagery. The capturing probability is sensitive to self-similar structure so that generated discrete pattern specifies the totality of most probable attractor points. Through simulation studies, extracted discrete patterns have been verified to maintain sufficient information to regenerate observed attractors.

References

[1] M. F. Barnsley. *Fractals Everywhere*, Academic Press, 1993.

[2] J. E. Hutchinson. Fractals and self similarity. *Indiana University Mathematical Journal*, 30:713–747, 1981.

[3] A. G. Jones and C. J. Taylor. Solving inverse problems in computer vision by scale space reconstruction. In *Proc. IAPR-MVA'94*, pages 401–404, 1994.

[4] K. Kamejima. Self-similarity detection in noisy feature distribution. In *Proc. ISCIE-SSS'99*, pages 301–306, 2000.

[5] K. Kamejima. Generic representation of self-similarity via structure sensitive sampling of noisy imagery. *Electronic Notes in Theoretical Computer Science*, 46: `http://www.elsevier.nl./locate/entcs/volume46.html` `20 pages`, 2001.