Rotation-type input-output-relationships for Wigner distribution moments in fractional Fourier transform systems

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ABSTRACT

It is shown how all global Wigner distribution moments of arbitrary order in the output plane of a (generally anamorphic) two-dimensional fractional Fourier transform system can be expressed in terms of the moments in the input plane. This general input-output relationship is then broken down into a number of rotation-type input-output relationships between certain combinations of moments. As an important by-product we get a number of moment combinations that are invariant under (anamorphic) fractional Fourier transformation.

1 Introduction

After the introduction of the Wigner distribution [1] (WD) for the description of coherent and partially coherent optical fields [2], it became an important tool for optical signal/image analysis and beam characterization [3, 4]. In this paper we show how the general relationship that relates the WD moments of arbitrary order in the input plane and the output plane of a (generally anamorphic) fractional Fourier transform (FT) system, can be broken down into a number of rotation-type relationships between certain combinations of the moments. Because of the rotation-type character of these latter relationships, they lead immediately to a number of moment combinations that are invariant under anamorphic fractional Fourier transformation.

2 Wigner distribution

The Wigner distribution of a two-dimensional function $f(x,y)$ is defined by

$$W_f(x,u;y,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+\tfrac{1}{2}x', y+\tfrac{1}{2}y') \times\ f^*(x-\tfrac{1}{2}x', y-\tfrac{1}{2}y')\exp[-j2\pi(ux'+vy')]dx'dy'. \tag{1}$$

The WD $W_f(x,u;y,v)$ represents a space function $f(x,y)$ in a combined space/spatial-frequency domain, the so-called phase space, where u is the spatial-frequency variable associated to the space variable x, and v the spatial-frequency variable associated to the space variable y. We remark that the definition of the WD – and all the results of this paper – need not be restricted to coherent light, in which case $f(x,y)$ would represent the complex field amplitude of the light, but can be extended to partially coherent light, in which case the two-point correlation function of the light can be identified with $< f(x+\frac{1}{2}x', y+\frac{1}{2}y')f^*(x-\frac{1}{2}x', y-\frac{1}{2}y') >$.

In this paper we consider the normalized moments of the WD, where the normalization is with respect to the total energy E of the signal:

$$E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)dxdudydv. \tag{2}$$

These normalized moments μ_{pqrs} of the WD are thus defined by

$$\mu_{pqrs}E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v) \times\ x^p u^q y^r v^s dxdudydv \quad (p,q,r,s \geq 0). \tag{3}$$

3 Fractional Fourier transform and moments in the fractional domain

The (anamorphic) fractional Fourier transform of a function $f(x,y)$ is defined by [5, 6, 7, 8, 9]

$$F_{\alpha\beta}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_\alpha(x,u)K_\beta(y,v)f(x,y)dxdy, \tag{4}$$

where the kernel $K_\alpha(x,u)$ is given by

$$K_\alpha(x,u) = \frac{\exp(j\frac{1}{2}\alpha)}{\sqrt{j\sin\alpha}}\exp\left[j\pi\frac{(x^2+u^2)\cos\alpha - 2ux}{\sin\alpha}\right]. \tag{5}$$

We remark that $F_{0,0}(u,v) = f(u,v)$ represents the function itself, while $F_{\pi/2,\pi/2}(u,v)$ corresponds to the normal two-dimensional FT of the function $f(x,y)$.

The fractional FT can be generated optically by a very simple, anamorphic, coherent-optical set-up, consisting only of two cylindrical lenses, whose focal lengths – in combination with some appropriate sections of free space – are related to the angles α and β.

One of the most important properties of the fractional FT is that it corresponds to a rotation of the WD in phase space:

$$W_{F_{\alpha\beta}}(x,u;y,v) = W_f(x\cos\alpha - u\sin\alpha, x\sin\alpha + u\cos\alpha; y\cos\beta - v\sin\beta, y\sin\beta + v\cos\beta). \quad (6)$$

We can as well define normalized moments $\mu_{pqrs}(\alpha,\beta)$ in the fractional domain and relate these to the original moments $\mu_{pqrs} = \mu_{pqrs}(0,0)$, cf. Eq. (3):

$$\mu_{pqrs}(\alpha,\beta)E = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x,u;y,v) \times\ x^p u^q y^r v^s dxdudydv \quad (p,q,r,s \geq 0) \quad (7)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)(x\cos\alpha + u\sin\alpha)^p \times\ (-x\sin\alpha + u\cos\alpha)^q (y\cos\beta + v\sin\beta)^r \times\ (-y\sin\beta + v\cos\beta)^s dxdudydv.$$

Note that the total energy E, see Eq. (2), is invariant under fractional Fourier transformation.

From the definitions of the normalized moments in the fractional domain, it follows directly that the $x^p u^q$ moments (with $r = s = 0$) are not affected by the fractional Fourier transformation in the y-direction (with angle β), while the $y^r v^s$ moments (with $p = q = 0$) are not affected by the one in the x-direction (with angle α):

$$\begin{aligned} \mu_{pq00}(\alpha,\beta) &= \mu_{pq00}(\alpha,0), \\ \mu_{00rs}(\alpha,\beta) &= \mu_{00rs}(0,\beta). \end{aligned} \quad (8)$$

4 Relations between moments in the fractional domain

To find simple relationships between the moments of $W_f(x,u;y,v)$ on the one hand and those of $W_{F_{\alpha,\beta}}(x,u;y,v)$ on the other, cf. Eq. (7), we define [10]

$$\begin{aligned} \xi &= x + ju, \\ \eta &= y + jv, \end{aligned} \quad (9)$$

$$\begin{aligned} \xi(\alpha) &= \exp(-j\alpha)\xi, \\ \eta(\beta) &= \exp(-j\beta)\eta, \end{aligned} \quad (10)$$

and

$$\begin{aligned} \xi_{2k,l}(\alpha) &= \xi(\alpha)^{k+l}\xi^*(\alpha)^k &= |\xi(\alpha)|^{2k}\xi(\alpha)^l \\ \eta_{2m,n}(\beta) &= \eta(\beta)^{m+n}\eta^*(\beta)^n &= |\eta(\beta)|^{2m}\eta(\alpha)^n, \end{aligned} \quad (11)$$

with $k,l,m,n \geq 0$. With $\xi_{2k,l}(0) = \xi_{2k,l}$ and $\eta_{2m,n}(0) = \eta_{2m,n}$, we then have

$$\begin{aligned} \xi_{2k,l}(\alpha)\eta_{2m,n}(\beta) &= \exp[-j(l\alpha + n\beta)]\xi_{2k,l}\eta_{2m,n}, \\ \xi_{2k,l}(\alpha)\eta^*_{2m,n}(\beta) &= \exp[-j(l\alpha - n\beta)]\xi_{2k,l}\eta^*_{2m,n}, \end{aligned} \quad (12)$$

which equations are equivalent to the rotation operators

$$\begin{bmatrix} \Re\{\xi_{2k,l}(\alpha)\eta_{2m,n}(\beta)\} \\ \Im\{\xi_{2k,l}(\alpha)\eta_{2m,n}(\beta)\} \end{bmatrix} = \mathbf{R}(l\alpha + n\beta)\begin{bmatrix} \Re\{\xi_{2k,l}\eta_{2m,n}\} \\ \Im\{\xi_{2k,l}\eta_{2m,n}\} \end{bmatrix},$$

$$\begin{bmatrix} \Re\{\xi_{2k,l}(\alpha)\eta^*_{2m,n}(\beta)\} \\ \Im\{\xi_{2k,l}(\alpha)\eta^*_{2m,n}(\beta)\} \end{bmatrix} = \mathbf{R}(l\alpha - n\beta)\begin{bmatrix} \Re\{\xi_{2k,l}\eta^*_{2m,n}\} \\ \Im\{\xi_{2k,l}\eta^*_{2m,n}\} \end{bmatrix}, \quad (13)$$

respectively, where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote the real and the imaginary part, respectively, and where $\mathbf{R}(\alpha)$ represents the rotation matrix

$$\mathbf{R}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}. \quad (14)$$

Relationships between the moments in the fractional (α,β) domain and the moments in the original domain follow from the relation [cf. Eq. (7)]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{F_{\alpha\beta}}(x,u;y,v)\xi_{2k,l}\eta_{2m,n}\, dxdudydv$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v)\xi_{2k,l}(\alpha)\eta_{2m,n}(\beta)\, dxdudydv$$
$$= \exp[-j(l\alpha + n\beta)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_f(x,u;y,v) \times \xi_{2k,l}\eta_{2m,n} dxdudydv \quad (15)$$

(and, if necessary, i.e., in the case that $l \neq 0$ and $n \neq 0$, from a similar relation based on $\xi_{2k,l}\eta^*_{2m,n}$), together with the detailed expressions for the real and imaginary parts of $\xi_{2k,l}\eta_{2m,n}$ (and of $\xi_{2k,l}\eta^*_{2m,n}$, if necessary). All relevant moment combinations up to fourth order, i.e., $2k + l + 2m + n \leq 4$, have been represented in Table 1, where we find the two vector entries t and b, and the corresponding rotation angle $l\alpha \pm n\beta$ [cf. Eq. (13)] for all different cases.

4.1 First-order moments

For the first-order moments we get 2 sets of equations, cf. Table 1: 2 equations for the x and u moments,

$$\begin{bmatrix} \mu_{1000}(\alpha,\beta) \\ \mu_{0100}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(\alpha)\begin{bmatrix} \mu_{1000} \\ \mu_{0100} \end{bmatrix}, \quad (16)$$

and 2 equations for the y and v moments,

$$\begin{bmatrix} \mu_{0010}(\alpha,\beta) \\ \mu_{0001}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(\beta)\begin{bmatrix} \mu_{0010} \\ \mu_{0001} \end{bmatrix}, \quad (17)$$

which constitute 4 equations for the 4 first-order moments.

Note that the following two expressions

$$\mu^2_{1000} + \mu^2_{0100}, \quad (18)$$
$$\mu^2_{0010} + \mu^2_{0001}, \quad (19)$$

are invariant under fractional Fourier transformation.

4.2 Second-order moments

For the second-order moments we get 3 sets of equations, cf. Table 1: 3 equations for the x and u moments,

$$\mu_{2000}(\alpha,\beta)+\mu_{0200}(\alpha,\beta)=\mu_{2000}+\mu_{0200},$$

$$\begin{bmatrix} \mu_{2000}(\alpha,\beta)-\mu_{0200}(\alpha,\beta) \\ 2\mu_{1100}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(2\alpha)\begin{bmatrix} \mu_{2000}-\mu_{0200} \\ 2\mu_{1100} \end{bmatrix}, \quad (20)$$

4 equations for the mixed moments,

$$\begin{bmatrix} \mu_{1010}(\alpha,\beta)-\mu_{0101}(\alpha,\beta) \\ \mu_{1001}(\alpha,\beta)+\mu_{0110}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(\alpha+\beta)\begin{bmatrix} \mu_{1010}-\mu_{0101} \\ \mu_{1001}+\mu_{0110} \end{bmatrix},$$

$$\begin{bmatrix} \mu_{1010}(\alpha,\beta)+\mu_{0101}(\alpha,\beta) \\ -\mu_{1001}(\alpha,\beta)+\mu_{0110}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(\alpha-\beta)\begin{bmatrix} \mu_{1010}+\mu_{0101} \\ -\mu_{1001}+\mu_{0110} \end{bmatrix}, \quad (21)$$

and 3 equations for the y and v moments,

$$\mu_{0020}(\alpha,\beta)+\mu_{0002}(\alpha,\beta)=\mu_{0020}+\mu_{0002},$$

$$\begin{bmatrix} \mu_{0020}(\alpha,\beta)-\mu_{0002}(\alpha,\beta) \\ 2\mu_{0011}(\alpha,\beta) \end{bmatrix} = \mathbf{R}(2\beta)\begin{bmatrix} \mu_{0020}-\mu_{0002} \\ 2\mu_{0011} \end{bmatrix}, \quad (22)$$

which constitute 3+4+3=10 equations for the 10 second-order moments. Note that we have the following 2+2+2=6 invariants, which can be directly derived from Table 1 (or from these 3 sets of equations):

$$\begin{aligned} \mu_{2000} &+ \mu_{0020}, && (23)\\ (\mu_{2000}-\mu_{0200})^2 &+ 4\mu_{1100}^2, && (24)\\ (\mu_{1010}-\mu_{0101})^2 &+ (\mu_{1001}+\mu_{0110})^2, && (25)\\ (\mu_{1010}+\mu_{0101})^2 &+ (-\mu_{1001}+\mu_{0110})^2, && (26)\\ \mu_{0020} &+ \mu_{0002}, && (27)\\ (\mu_{0020}-\mu_{0002})^2 &+ 4\mu_{0011}^2. && (28) \end{aligned}$$

4.3 Higher-order moments

For higher-order moments we can proceed analogously, using the expressions given in Table 1. For the third-order moments we get 4 sets of equations, yielding 4+6+6+4=20 equations for 20 variables, and 2+3+3+2=10 invariant combinations of third-order moments. For the fourth-order moments we get 5 sets of equations, yielding 5+8+9+8+5=35 equations for 35 variables, and 3+4+5+4+3=19 invariant combinations of fourth-order moments.

5 Conclusions

We have shown how the general relationship that relates the WD moments of arbitrary order in the input plane and the output plane of an anamorphic fractional Fourier transform (FT) system, can be broken down into a number of rotation-type relationships between certain combinations of the moments. Because of the rotation-type character of these latter relationships, they lead immediately to a number of moment combinations that are invariant under anamorphic fractional Fourier transformation.

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Table 1: Moment combinations undergoing a rotation of the form of Eqs. (13) up to fourth order

$$\left[\begin{array}{c} t(\alpha,\beta) \\ b(\alpha,\beta) \end{array}\right] = \mathbf{R}(l\alpha \pm n\beta) \left[\begin{array}{c} t(0,0) \\ b(0,0) \end{array}\right] = \left[\begin{array}{cc} \cos(l\alpha \pm n\beta) & \sin(l\alpha \pm n\beta) \\ -\sin(l\alpha \pm n\beta) & \cos(l\alpha \pm n\beta) \end{array}\right] \left[\begin{array}{c} t(0,0) \\ b(0,0) \end{array}\right]$$

top vector entry t	bottom vector entry b	angle $l\alpha \pm n\beta$
μ_{1000}	μ_{0100}	α
μ_{0010}	μ_{0001}	β
$\mu_{2000}+\mu_{0200}$		
$\mu_{2000}-\mu_{0200}$	$2\mu_{1100}$	2α
$\mu_{1010}-\mu_{0101}$	$\mu_{1001}+\mu_{0110}$	$\alpha+\beta$
$\mu_{1010}+\mu_{0101}$	$-\mu_{1001}+\mu_{0110}$	$\alpha-\beta$
$\mu_{0020}+\mu_{0002}$		
$\mu_{0020}-\mu_{0002}$	$2\mu_{0011}$	2β
$\mu_{3000}+\mu_{1200}$	$\mu_{2100}+\mu_{0300}$	α
$\mu_{3000}-3\mu_{1200}$	$3\mu_{2100}-\mu_{0300}$	3α
$\mu_{2010}+\mu_{0210}$	$\mu_{2001}+\mu_{0201}$	β
$\mu_{2010}-\mu_{0210}-2\mu_{1101}$	$\mu_{2001}-\mu_{0201}+2\mu_{1110}$	$2\alpha+\beta$
$\mu_{2010}-\mu_{0210}+2\mu_{1101}$	$-\mu_{2001}+\mu_{0201}+2\mu_{1110}$	$2\alpha-\beta$
$\mu_{1020}+\mu_{1002}$	$\mu_{0120}+\mu_{0102}$	α
$\mu_{1020}-\mu_{1002}-2\mu_{0111}$	$2\mu_{1011}+\mu_{0120}-\mu_{0102}$	$\alpha+2\beta$
$\mu_{1020}-\mu_{1002}+2\mu_{0111}$	$-2\mu_{1011}+\mu_{0120}-\mu_{0102}$	$\alpha-2\beta$
$\mu_{0030}+\mu_{0012}$	$\mu_{0021}+\mu_{0003}$	β
$\mu_{0030}-3\mu_{0012}$	$3\mu_{0021}-\mu_{0003}$	3β
$\mu_{4000}+2\mu_{2200}+\mu_{0400}$		
$\mu_{4000}-\mu_{0400}$	$2\mu_{3100}+2\mu_{1300}$	2α
$\mu_{4000}-6\mu_{2200}+\mu_{0400}$	$4\mu_{3100}-4\mu_{1300}$	4α
$\mu_{3010}+\mu_{1210}-\mu_{2101}-\mu_{0301}$	$\mu_{3001}+\mu_{1201}+\mu_{2110}+\mu_{0310}$	$\alpha+\beta$
$\mu_{3010}+\mu_{1210}+\mu_{2101}+\mu_{0301}$	$-\mu_{3001}-\mu_{1201}+\mu_{2110}+\mu_{0310}$	$\alpha-\beta$
$\mu_{3010}-3\mu_{1210}-3\mu_{2101}+\mu_{0301}$	$\mu_{3001}-3\mu_{1201}+3\mu_{2110}-\mu_{0310}$	$3\alpha+\beta$
$\mu_{3010}-3\mu_{1210}+3\mu_{2101}-\mu_{0301}$	$-\mu_{3001}+3\mu_{1201}+3\mu_{2110}-\mu_{0310}$	$3\alpha-\beta$
$\mu_{2020}+\mu_{2002}+\mu_{0220}+\mu_{0202}$		
$\mu_{2020}-\mu_{2002}+\mu_{0220}-\mu_{0202}$	$2\mu_{2011}+2\mu_{0211}$	2β
$\mu_{2020}+\mu_{2002}-\mu_{0220}-\mu_{0202}$	$2\mu_{2120}+2\mu_{1102}$	2α
$\mu_{2020}-\mu_{2002}-\mu_{0220}+\mu_{0202}-4\mu_{1111}$	$2\mu_{2011}-2\mu_{0211}+2\mu_{1120}-2\mu_{1102}$	$2\alpha+2\beta$
$\mu_{2020}-\mu_{2002}-\mu_{0220}+\mu_{0202}+4\mu_{1111}$	$-2\mu_{2011}+2\mu_{0211}+2\mu_{1120}-2\mu_{1102}$	$2\alpha-2\beta$
$\mu_{1030}+\mu_{1012}-\mu_{0121}-\mu_{0103}$	$\mu_{1201}+\mu_{1003}+\mu_{0130}+\mu_{0112}$	$\alpha+\beta$
$\mu_{1030}+\mu_{1012}+\mu_{0121}+\mu_{0103}$	$-\mu_{1201}-\mu_{1003}+\mu_{0130}+\mu_{0112}$	$\alpha-\beta$
$\mu_{1030}-3\mu_{1012}-3\mu_{0121}+\mu_{0103}$	$3\mu_{1201}-\mu_{1003}+\mu_{0130}-3\mu_{0112}$	$\alpha+3\beta$
$\mu_{1030}-3\mu_{1012}+3\mu_{0121}-\mu_{0103}$	$-3\mu_{1201}+\mu_{1003}+\mu_{0130}-3\mu_{0112}$	$\alpha-3\beta$
$\mu_{0040}+2\mu_{0022}+\mu_{0004}$		
$\mu_{0040}-\mu_{0004}$	$2\mu_{0031}+2\mu_{0013}$	2β
$\mu_{0040}-6\mu_{0022}+\mu_{0004}$	$4\mu_{0031}-4\mu_{0013}$	4β