QUATERNION SUBSPACE METHOD FOR VECTOR-SENSOR WAVE SEPARATION

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ABSTRACT

In this paper, we propose a new approach for vector-sensor signal modeling and processing. We introduce the way quaternions allows to characterize signals collected on vector-sensor array. In physics or geophysics, vector signals contain several informations, like waveform (or source wavelet) and polarization. In order to access to the physical informations carried by the different wave fields recorded, it is necessary to develop wave or source separation techniques. So, we give here the extension to the quaternion case of a widely used tool in signal processing : the Singular Value Decomposition. We expose the way it can be computed, and develop a quaternionic subspace method for polarized wave separation using this new tool based on quaternion matrix algebra. We finally show on synthetic data the potentiality of this new quaternionic signal processing technique.

1 Vector-sensor array

The use of vector-sensor array has been developed in many areas such as electromagnetic, seismic, communications. The vector-sensor (or multicomponent sensor) record vibrations in one, two or three directions of space.

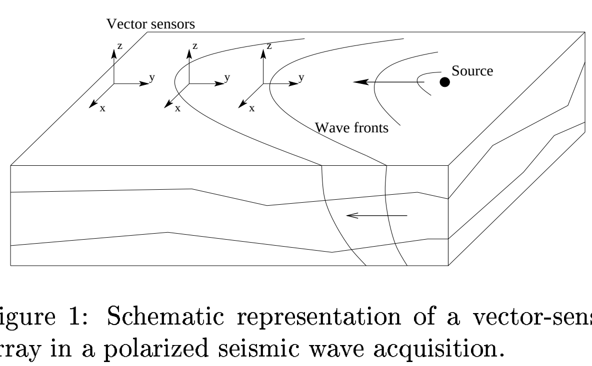

Figure 1: Schematic representation of a vector-sensor array in a polarized seismic wave acquisition.

This is performed using dipoles for electromagnetic waves and directional geophones for seismic waves. A schematic representation of a vector-sensor array is given on figure 1. Using this kind of arrays, we can access to a physical property of waves : polarization. It represents the phase and amplitude relations between the signals recorded on the components of a vector-sensor. Classically, we can find three kind of polarizations (elliptical, linear and circular) depending on the type of wave recorded on the array. We propose here to use quaternions to encode and process polarized signals, in order to involve polarization in the processing of these signals.

1.1 Quaternions

Quaternions, discovered in 1843 by Hamilton W.R. [1], are the extension of complex numbers to 3D space. A quaternion is composed of a real part and three imaginary parts:

$$q = a + bi + cj + dk \tag{1}$$

where

$$\begin{gathered} i^2 = j^2 = k^2 = -1, \\ ij = k \ , \ ji = -k \end{gathered} \tag{2}$$

The conjugate $\overline{q}$ of a quaternion can be written as: $\overline{q} = a - bi - cj - dk$. A pure quaternion is a quaternion which real part is null ($a = 0$). The norm of a quaternion is $|q| = \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{a^2 + b^2 + c^2 + d^2}$ and its inverse $q^{-1} = \frac{\overline{q}}{|q|}$. The most particular characteristic of quaternions is that they are non-commutative under multiplication. Given two quaternions q_1 and q_2, we have the inequality $q_1 q_2 \neq q_2 q_1$. This property involves that for the model and processing of quaternion signals, it is necessary to work on a right vector space. In order to develop new tools for quaternion signals, we now introduce the vector space these signals lie in.

1.2 Quaternion Hilbert space

We introduce the basic definitions needed for quaternionic signal processing.

1.2.1 Quaternion vector

Classically, in numerical signal processing, signals are represented in an Hilbert space. This can be also done for quaternionic signal, by working on a quaternion Hilbert space. This quaternion Hilbert space is a right vector space, *i.e.* scalar are multiplied on the right in a scalar-vector product. In this Hilbert space, a quaternionic signal of N samples is a vector which elements are quaternions: $\mathbf{x} = [x_1\ x_2\ ...\ x_N]^T \in \mathbb{H}^N$. We then define a scalar product and a norm on this vector space as:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{N} \overline{x}_i y_i \quad \text{and} \quad \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \tag{3}$$

A metric must also be defined in order to measure the distance between two quaternion vectors $\mathbf{x}$ and $\mathbf{y}$:

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = [(\mathbf{x} - \mathbf{y})^{\triangleleft}(\mathbf{x} - \mathbf{y})]^{\frac{1}{2}} \tag{4}$$

where $\triangleleft$ represents the quaternionic transposition-conjugate operator. These tools allow to manipulate quaternion signals as vectors on this quaternion Hilbert space.

1.2.2 Quaternion matrix

Given a set of M signals from $\mathbb{H}^N$, it can be arranged in a matrix $\mathbf{S} = \left[\mathbf{x}_1^T\ \mathbf{x}_2^T\ ...\ \mathbf{x}_M^T\right]$. This matrix is quaternion valued and defines a vector space : $\mathbf{S} \in \mathbb{H}^{N \times M}$. This definition will now allow us to express the set of signals recorded on a vector-sensor array using a quaternion matrix.

1.3 Vector-sensor signal model

Signals recorded on vector-sensor array are often arranged in a long-vector form [5] before processing, or sometimes, processed component-wise. The use of long-vectors may be restrictive if all the possible way to build this vector are not examined [4]. A consequence of processing the components separately is that the relations between the signals they record are not taken into account. In order to process simultaneously the all dataset and to preserve the polarization relations, we propose to encode vector-sensor signals as quaternion signals. The model proposed here is close to the one given by Sangwine [6] to represent colour images, where the three imaginary parts of a pure quaternion are used to represent the red, blue and green components of a pixel.

In a same way, assuming an array composed of N_x vector-sensors, each of them having three orthogonal components recording N_t time samples, we can write the set of signals recorded on this array like :

$$\mathbf{S} = \begin{bmatrix} x_1(t)i + y_1(t)j + z_1(t)k \\ x_2(t)i + y_2(t)j + z_2(t)k \\ \vdots \\ x_{N_x}(t)i + y_{N_x}(t)j + z_{N_x}(t)k \end{bmatrix} = \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_{N_x}(t) \end{bmatrix} \tag{5}$$

where the quaternion signal at n^{th} sensor is :

$$s_n(t) = x_n(t)i + y_n(t)j + z_n(t)k \tag{6}$$

where $x_n(t)$, $y_n(t)$ and $z_n(t)$ are the signals recorded on the three component of this sensor. Doing so, the set of signals recorded on the vector-sensor array, $\mathbf{S}$, is a matrix of size $N_x \times N_t$ which elements are pure quaternions (*i.e.* $\mathbf{S} \in \mathbb{H}^{N_x \times N_t}$). With this model, vector signals processing can be obtained by extension of classical real or complex algorithms to their quaternion case. In order to do so, we now introduce some elements of quaternion matrix algebra which are the basics needed for vector signal processing.

2 Quaternion matrix algebra

We present here a few concepts of quaternion matrix algebra that allow to manipulate a set of quaternion signals.

2.1 Cayley-Dickson Notation

Given a quaternion q (eq. 1), we can rewrite it as :

$$q = \alpha + \beta j \tag{7}$$

where α and β are complex numbers given as : $\alpha = a + ib$ and $\beta = c + id$. Some theorems known for real and complex case can easily be extended to the quaternion case using this notation. The expression of matrix operations (and their computation) on quaternion field is possible using the isomorphism that exists between quaternion algebra and complex algebra. Some complex notations are though needed to rewrite vectors and matrices in complex form.

2.1.1 Complex representation of a quaternion vector

A quaternion vector $\mathbf{x} \in \mathbb{H}^N$ can be expressed using the Cayley-Dickson notation:

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2\, j \tag{8}$$

with $\mathbf{x}_1$ and $\mathbf{x}_2 \in \mathbb{C}^N$. We can then define a bijection $f : \mathbb{H}^N \to \mathbb{C}^{2N}$

$$f(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_1 \\ -\overline{\mathbf{x}}_2 \end{bmatrix} \tag{9}$$

Linear properties are invariant under this bijection.

2.1.2 Complex representation of a quaternion matrix

Given a matrix of quaternions $\mathbf{S} \in \mathbb{H}^{N \times M}$, its expression using Cayley-Dickson notation is:

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2\, j \tag{10}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are complex matrices (*i.e.* $\in \mathbb{C}^{N \times M}$). We can then define the *complex adjoint* matrix, noted $\chi_S \in \mathbb{C}^{2N \times 2M}$, corresponding to the quaternion matrix $\mathbf{S}$ like:

$$\chi_S = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ -\overline{\mathbf{S}}_2 & \overline{\mathbf{S}}_1 \end{pmatrix} \tag{11}$$

Some properties of χ_S are given in [8]. This complex notation for quaternion matrices can be used to compute quaternion matrix decomposition using complex decomposition algorithm.

2.2 SVD of a quaternion matrix

Any matrix $\mathbf{S} \in \mathbb{H}^{N \times M}$ admits a Singular Value Decomposition [9] given as:

$$\mathbf{S} = \mathbf{U} \begin{pmatrix} \mathbf{\Sigma}_r & 0 \\ 0 & 0 \end{pmatrix} \mathbf{V}^{\triangleleft} \tag{12}$$

where $\mathbf{U} \in \mathbb{H}^{N \times N}$ and $\mathbf{V} \in \mathbb{H}^{M \times M}$ are unitary quaternion matrices. These matrices contain the left and right quaternionic singular vectors of $\mathbf{S}$. $\mathbf{\Sigma_r}$ is a real diagonal matrix, where r is the rank of $\mathbf{S}$ (*i.e.* the number of non-null singular values). The singular elements of a quaternion matrix are obtained from the SVD of the *complex adjoint* matrix χ_S (see [3]). Left and right singular vectors of $\mathbf{S}$ are related to the left and right singular vectors of χ_S by the bijection f defined above (Eq. 8). Due to its structure (Eq. 11), the singular values of χ_S are by pairs and are equal to the ones of $\mathbf{S}$ [3]. The computation of the SVD for a $N \times M$ quaternion matrix is equivalent to the computation of the SVD of a $2N \times 2M$ complex matrix (its complex adjoint) and so can be achieved using classical algorithms [2].

3 Quaternionic subspace method

A direct and simple method to process wave separation on a 2D dataset, is to decompose the original vector space defined by the data in two orthogonal subspaces, one called the "signal subspace" and the other one called the "noise subspace". This well known technique is based on the SVD of the original dataset and can be extended to vector-sensor 2D dataset using the Quaternionic SVD presented above.

3.1 Subspaces on quaternion vector space

As in real or complex case, it is possible to decompose a quaternion vector space into two orthogonal subspaces. This decomposition is directly obtained from the SVD of the matrix considered. The first subspace is built with the highest singular values and their corresponding singular vectors. A quaternion vector space $< \mathbf{S} >$ can be decomposed as:

$$< \mathbf{S} > = < \mathbf{S}_1 > \oplus < \mathbf{S}_2 > \tag{13}$$

where the subspace $< \mathbf{S}_1 >$ is a rank α truncation of the Singular Value Decomposition of $\mathbf{S}$ (see [7]) and $\oplus$ denotes a direct sum of vector spaces. $< \mathbf{S}_1 >$ and $< \mathbf{S}_2 >$ define the orthogonal decomposition of the original vector space defined by $\mathbf{S}$. Similarly to real and complex case, the rank r truncation of the SVD of $\mathbf{S}$ is the *Best rank r approximation* of the matrix $\mathbf{S}$. The orthogonality between the two subspaces is now going to be exploited to develop a wave separation technique for vector-sensor 2D array.

3.2 Wave separation

The original quaternion dataset collected on a vector-sensor array, noted $\mathbf{S}$, can be decomposed in two subspaces, denoted "signal" and "noise":

$$\mathbf{S} = \mathbf{S}_{signal} + \mathbf{S}_{noise} \tag{14}$$

The SVD of $\mathbf{S}$ can be first rewritten as:

$$\mathbf{S} = \sum_{n=1}^{r} \mathbf{u}_n \mathbf{v}^{\triangleleft}_n \lambda_n \tag{15}$$

and so the expression of the two subspaces is:

$$\mathbf{S} = \sum_{n=1}^{\alpha} \mathbf{u}_n \mathbf{v}^{\triangleleft}_n \lambda_n + \sum_{m=\alpha+1}^{r} \mathbf{u}_m \mathbf{v}^{\triangleleft}_m \lambda_m \tag{16}$$

where r is the rank of the dataset $\mathbf{S}$. $\mathbf{u}_n$ (respectively $\mathbf{v}$) is the n^{th} left (right) singular vector of $\mathbf{S}$ and λ_n is the associated singular value. The Quaternionic SVD is a canonical decomposition of the original polarized dataset, that express $\mathbf{S}$ as a sum of r rank 1 matrices or *singular matrices*. The signal subspace is then built with α *singular matrices* and the noise subspace with the $r - \alpha$ others. The physical information contained in the different parts of the SVD can be identified as:

- $\mathbf{v}_n$ (columns of $\mathbf{V}$): singular vectors that define an orthogonal basis of "singular polarized seismic wavelets".
- $\mathbf{u}_n$ (columns of $\mathbf{U}$): singular vectors that define an orthogonal basis of "singular behavior" of the seismic wavelets on the vector array.
- λ_n : magnitude associated to the n^{th} wavelet.

This identification can help to identify the waves estimated by this technique and may be useful to validate the physical characteristics of the estimated waves.

4 Application on synthetic data

To evaluate the Quaternionic SVD potentiality, we simulate a dataset collected by an array of 10 vector-sensors ($N_x = 10$). Each sensor has got three components. Each signal is composed of 128 samples ($N_t = 128$). On figure 2, we present the three components of the simulated polarized seismic wave. This wave is dispersive, meaning that there is a constant phase shift between the signals recorded on the 10 sensors. The polarization is elliptical, *i.e.* there are amplitude and phase relations between the three components.

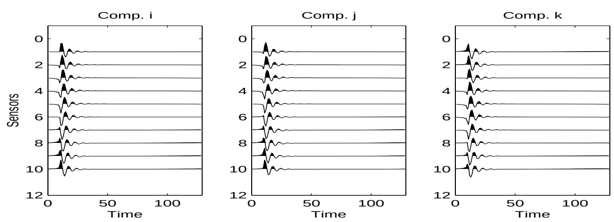

Figure 2: Three components of the original wave. The wave is dispersive and its polarization is elliptical.

White noise is then added to the polarized wave (Fig. 4). The mixture is then encoded using pure quaternions (Eq. 5).

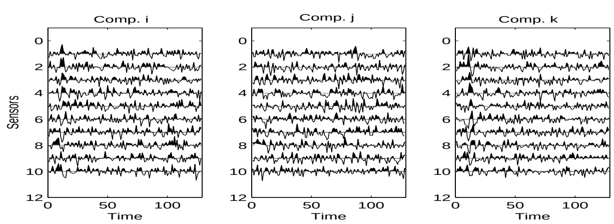

Figure 3: Three components of the mixture composed of the elliptically polarized wave and the noise.

The Quaternionic SVD is used to isolate the wave from the noise, under the assumption that noise and signal are uncorrelated. We used only one *singular matrix* to build the signal subspace (fig. 4).

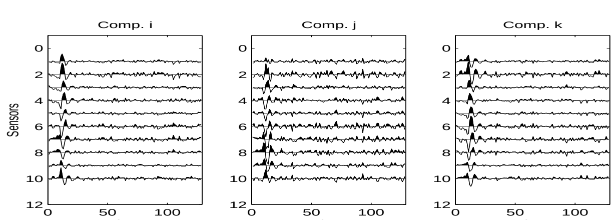

Figure 4: Three components of the estimated "signal subspace" by rank 1 truncation of the quaternionic SVD.

With this rank 1 signal subspace, and thanks to the quaternion model, we recover the wave form, the polarization and the dispersive parameters in a single step, which is not feasible using real SVD separately on the three components. Quaternion model increase the separation robustness to noise and takes into account the polarization parameters as well as any phase shift information (like dispersion). This show that the Quaternionic SVD is a powerful tool to process vector-sensor array signals and that the quaternion model of vector signals is a good way to handle the physical properties carried by these signals.

5 Conclusion

We have proposed a new model for vector-sensor array signals based on quaternion numbers and introduced the concepts of *quaternion signal* for vector-sensor arrays. This way to model vector signals takes into account the polarization of the waves in the processing step, so that physical information is exploited to improve the processing results. So, the quaternionic subspace method we propose gives better results than a component-wise subspace method would do. The Quaternionic SVD introduced here may be a useful tool for vector-signals processing. Finally, as polarized signals are frequently encountered in physics (electromagnetic, seismic, optics), the method proposed here could easily be transposed to all kind of polarized signals.

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