GENERALIZED DECISION ERROR PROBABILITY FOR LINEAR EQUALIZATION USING *M*-ARY PAM TRANSMISSION

Rainfield Y. Yen, Hong-Yu Liu and Jin-Qun Chen
Department of Electrical Engineering, Tamkang University
Tamsui, Taipei, Taiwan, R.O.C.
Tel: 886-2-2621-5656-X2734; fax: 886-2-2622-1565
e-mail: 059270@mail.tku.edu.tw

ABSTRACT

A generalized formula for decision error probability is derived in terms of equalizer tap weight coefficients for linear equalization in bandlimited channels employing M-ary PAM transmission. The generalized decision error probability is valid for any equalizer tap weight assignment (non-optimum or optimum state) as well as for any equalization algorithm. From the generalized expression, we then deduce the formulas for zero-forcing, MSE, and other equalizers at optimum state, i.e., when the equalizer is optimized.

Two examples are demonstrated. Decision error probabilities are obtained for these channels theoretically using the derived formulas as well as by Monte Carlo simulations. It is found that simulation results agree excellently with the theoretical formulas.

1 INTRODUCTION

Extensive works have been devoted in the past to the obtaining of decision error probabilities for linear equalizer systems [1-7]. Aaron and Tufts gave an error probability based on the conditional error probability concept [1]. Saltzberg derived an upper bound for the error probability using the worst-case sequence [2]. Lugannani evaluated an error bound by the Chernoff inequality [3]. Ho and Yeh evaluated the error probability in terms of the first $2k$ moments of the ISI [4]. Glave derived an error bound for correlated binary signals [5]. And Yao and Tobin obtained upper and lower bounds for the error probability from the theory of moment spaces [6]. Most of these results are either too complicated or tedious, or limited to special cases only (e.g., binary transmissions only or MSE case only), or just give the bounds only. In this work, a generalized formula of decision error probability is derived for any linear equalizer system (any algorithm) in bandlimited channels employing M-ary PAM transmission. The approach is close to but goes beyond that of Aaron and Tufts. The resultant mathematical expression will be in terms of equalizer tap weight coefficients and valid for any assignment of tap weights (non-optimum or optimum state).

From the generalized formula, we then deduce the formula for optimum state (i.e., when the linear equalizer is optimized) and apply it to zero-forcing, MSE, and other types of linear equalizers. The result shows that the additive channel noise is enhanced at the equalizer output by a factor equal to the energy in the equalizer tap weights.

Next, We present two case examples. First is a theoretical case of *RC* lowpass channel. The second example is a case of bandlimited channel with a discrete characteristic. For both cases, Monte Carlo simulations are performed for zero-forcing as well as MSE equalization, and simulation results are compared with theoretical formulas.

This paper is organized as follows. Section 2 derives the generalized decision error probability. Section 3 gives the decision error probability for optimized equalizers. We then treat a theoretical case of an *RC* lowpass channel in section 4. Then in section 5, we perform the same analysis for a typical telephone channel with a discrete-time characteristic. Monte Carlo simulations are performed for both cases and are found in good agreement with the theoretical findings. And finally, conclusions are drawn in section 6.

2 GENERALIZED PROBABILITY OF DECISION ERROR FOR DIGITAL SYSTEMS

We consider a bandlimited channel employing M-ary PAM transmission having the discrete response f_k, $k=-L,...,0,...,L$, and let $\sum_k f_k^2 = 1$ for normalization. A linear equalizer at the receiving end has tap weights w_k, $k=-N,...,0,...,N$. With input data x_k and channel noise h_k, the equalizer output estimate $\hat{x}_k$ is given by

$$\hat{x}_k = x_k * q_k + h_k * w_k, \tag{1}$$

where $q_k = f_k * w_k$ denotes the cascade of channel and equalizer, and h_k is a zero mean Gaussian random variable with variance $N_0/2$. The source data symbols $\{x_k\}_{-\infty}^{\infty}$ takes the discrete values with equal probability given by

$$x_k = (2m-1-M)d\sqrt{E_g}, \tag{2}$$

where $m=1,2,..,M$, $M = 2^n$ with n being the number of bits per symbol, and E_g is the energy of a transmitting pulse $g(t)$. It is easily shown that the average energy of x_k is $E_{av} = \frac{1}{3}(M^2-1)d^2E_g$ [8]. Taking $E_{av}=1$, we get

$$d = \sqrt{\frac{3}{(M^2-1)E_g}}. \tag{3}$$

It is also easy to show that $2d\sqrt{E_g}$ is the distance between adjacent data symbols. The estimate error e_k is

$$e_k = x_k - \hat{x}_k = (x_k - x_k * q_k) - h_k * w_k. \tag{4}$$

The last term on the right hand side of (4) $h_k * w_k$ is obviously a zero mean Gaussian random variable a variance of $\|\mathbf{w}\|^2 \; N_0/2$, where

$$\|\mathbf{w}\|^2 = \sum_{i=-N}^{N} w_i^2 \tag{5}$$

is the squared norm of the equalizer tap weight vector $\mathbf{w} = [w_{-N},\cdots,w_0,\cdots,w_N]^T$ with T denoting transpose.

Now, define z_k as

$$z_k = x_k - x_k * q_k. \tag{6}$$

Since x_k is a discrete uniformly distributed random variable with M possible outcomes, it is obvious from (6) that z_k is a discrete uniformly distributed random variable with $D = M^{2L+2N+1}$ possible outcomes, say, a_i, $i = 1, 2, \cdots, D$, with probability

$$P(z_k = a_i) = \frac{1}{D}, \quad i = 1, 2, \cdots, D. \tag{7}$$

For a set outcome of z_k, say a_i, the estimate error has a p.d.f. given by

$$p(e_k \mid a_i) = \frac{1}{\sqrt{pN_0}\,\|\mathbf{w}\|} \exp[-(e_k - a_i)^2 / \|\mathbf{w}\|^2 N_0]. \tag{8}$$

Therefore, the probability of equalizer decision error P_M for M-ary PAM transmission can now be readily obtained as follows:

$$\begin{aligned} P_M &= \frac{2(M-1)}{M} \int_{d\sqrt{E_g}}^{\infty} p(e_k) de_k \\ &= \frac{2(M-1)}{M} \int_{d\sqrt{E_g}}^{\infty} \frac{1}{D} \sum_{i=1}^{D} p(e_k \mid a_i) de_k \\ &= \frac{2(M-1)}{M} \cdot \frac{1}{D} \sum_{i=1}^{D} Q\left(\frac{d\sqrt{E_g} - a_i}{\|\mathbf{w}\| \sqrt{N_0/2}} \right). \end{aligned} \tag{9}$$

Equation (9) is in fact a generalized formula. It applies to any assignment of equalizer tap weights, whether the tap weights are in non-optimum condition or optimum condition. It is also valid for any algorithm, whether the algorithm is zero-forcing, MSE, or least square, or others. However, this formula is not quite informative, for the D values of a_i are nowhere to be determined. But, later, when we come to optimum conditions, this indeterminacy will disappear.

There are two limiting cases to be noted. First is when channel noise is zero. In this case, (4) and $p(e_k)$ respectively reduce to

$$e_k = z_k = x_k - \hat{x}_k = x_k - x_k * q_k,$$

and $$p(e_k) = \frac{1}{D} \sum_{i=1}^{D} d(e_k - a_i), \tag{10}$$

where we have used the fact that

$$\lim_{s \to 0} \frac{1}{\sqrt{2ps}} e^{-x^2/2s^2} = d(x) \tag{11}$$

As a result, the decision error probability becomes

$$P_M = \frac{2(M-1)}{M} \cdot \frac{K}{D}, \tag{12}$$

where K is the number of a_i 's in the set $\{a_1, a_2, \cdots, a_D\}$ that are greater than $d\sqrt{E_g}$.

Another limiting case is when $\|\mathbf{w}\| = 0$. In this case, $\hat{x}_k = 0$, and hence $e_k = x_k$, and D possible outcomes of z_k reduce to only M outcomes of x_k (i.e., $\pm d\sqrt{E_g}, \pm 3d\sqrt{E_g}, \cdots, \pm(M-1)d\sqrt{E_g}$). Hence, upon applying (10),

$$p(e_k) = \frac{1}{M} \sum_{m=1}^{M} d[e_k - (2m-1-M)d\sqrt{E_g}],$$

and $$P_M = \frac{2(M-1)}{M} \cdot \frac{1}{2} = \frac{M-1}{M}, \tag{13}$$

or the probability of correct decision is

$$P_C = 1 - P_M = \frac{1}{M}. \tag{14}$$

Out of M occasions, 2 occasions will result in correct decision, namely, when $x_k = \pm d\sqrt{E_g}$, each, when occurring, has a 50-50 chance of correct guessing by toss of a coin

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3 OPTMIZED EQUALIZERS

In the previous section, we have given a generalized expression for the decision error probability for linear equalization which can be applied to non-optimum as well as optimum state for any algorithm. But, in real practice, we will only be interested in the optimum state, i.e., when the equalizer is optimized. We now discuss this optimum situation for various algorithms.

3.1 Zero-forcing equalization

For zero-forcing equalization, assuming the equalizer has infinite length, then the equalizer will be an exact inverse filter to the channel [8]. The coefficient q_k satisfies the conditions $q_0 = 1$ and $q_k = 0$ for $k \neq 0$, and hence $\sum_{k \neq 0} q_k = 0$. With these conditions, (6) becomes $z_k = 0$, and hence $a_i = 0$ for all i. Then, along with (3), (9) becomes

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6}{\|\mathbf{w}_o\|^2 N_0 (M^2 - 1)}} \right). \tag{15}$$

Here, we have replaced $\|\mathbf{w}_o\|$ for $\|\mathbf{w}\|$ to indicate that the equalizer is now in optimum condition.

3.2 MSE equalization

For MSE equalization with infinite equalizer length, it is well known that [8]

$$W'(w) = \frac{F'^*(w)}{F'(w)F'^*(w) + N_0/2}, \tag{16}$$

where the superscript * denotes complex conjugate. Let $Q'(w)$ be the *DTFT* (discrete-time Fourier transform) of the sequence q_k, we know that

$$Q'(w) = F'(w)W'(w) = \frac{F'(w)F'^*(w)}{F'(w)F'^*(w) + N_0/2}. \tag{17}$$

For large received signal-to noise ratios (small N_0),

$$Q'(w) \cong 1, \text{ and } W'(w) \cong \frac{1}{F'(w)}. \tag{18}$$

Then, we get $q_0 \cong 1$, $q_k \cong 0$ for $k \neq 0$, and hence $\sum_{k \neq 0} q_k \cong 0$. This simply means that the MSE equalizer is very close to a zero-forcing equalizer. This fact is, of course, well known [8]. But it also means the intersymbol interference is almost completely eliminated by the equalizer. Using the same reasoning for zero-forcing equalization at optimum state, we can use (9) along with (3) to obtain an approximate decision error probability for MSE equalization at optimum state (for large SNR) as

$$P_M \cong \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6}{\|\mathbf{w}_o\|^2 N_0 (M^2 - 1)}} \right). \tag{19}$$

When SNR is not large, (19) should be less accurate but nonetheless still reasonably acceptable as will be shown later by simulations.

3.3 Other algorithms

For any algorithm, whether it is LMS (which is a stochastic gradient adaptive MSE algorithm), least-squares or RLS (which is a weighted least-squares adaptive algorithm), etc., it is clear that when SNR is large and the equalizer is in optimum condition, it should be quite much an inverse to the channel, which in turn, implies $q_0 \cong 1$, $q_k \cong 0$ for $k \neq 0$. Therefore, by the same argument given above, the decision error probability can be given by (19).

Notice that, neither (15) nor (19) contains the indeterminable terms a_i. Further notice that, when the additive channel noise is absent, it is easily seen that the K in (12) becomes zero resulting in zero decision error. In other words, without the presence of channel noise, a linear equalizer can achieve error-free decisions.

One more word needs be said about the norm $\|\mathbf{w}_o\|$. Notice that (15) or (19) closely resembles the error probability for M-ary PAM transmission in an infinite bandwidth channel (no ISI) [8] except for an additional term of $\|\mathbf{w}_o\|^2$ which is greater than 1 when SNR is large as shown below. This term accounts for the performance degradation caused by noise enhancement by the equalizer. For the MSE equalizer, with $Q'(w) = F'(w)W'(w) \cong 1$ at large SNR, applying Schwarz inequality on $|W'(w)|$ and $|F'(w)|$, we have

$$\left[\frac{1}{2p}\int_{-p}^{p}|W'(w)|^2\,dw\right]\cdot\left[\frac{1}{2p}\int_{-p}^{p}|F'(w)|^2\,dw\right] \geq \left[\frac{1}{2p}\int_{-p}^{p}|W'(w)F'(w)|\,dw\right]^2 \cong 1$$

, or, since $|W'(w)| \cong |1/F'(w)| \neq |F'(w)|$, we can exclude the possibility of equality to get

$$\|\mathbf{w}_o\|^2 \cdot \|\mathbf{f}\|^2 > 1. \qquad (20)$$

Since we have $\|\mathbf{f}\|^2 = \sum_k f_i^2 = 1$. Thus

$$\|\mathbf{w}_o\|^2 > 1 \text{ or, } \|\mathbf{w}_o\| > 1. \qquad (21)$$

This is to be expected since a performance with ISI cannot be better than without ISI.

However, simulation results show that, when SNR is small, (21) becomes untrue and the conditions $q_0 \cong 1$, $q_k \cong 0$ for $k \neq 0$ are also violated. When $N_0 \to \infty$, it can be seen from (16) and (17) that both $W'(w)$ and $Q'(w)$ approach zero, thus (18) holds no more. However, from computer simulations, if SNR is greater than about 20 dB, (19) will be acceptable.

4 THEORETICAL *RC* LOWPASS CHANNEL

We will now discuss a theoretical case, namely, a channel that possesses an impulse response of an RC lowpass filter

$$c(t) = \frac{1}{RC}e^{-t/\mathrm{RC}} \qquad t \geq 0. \qquad (22)$$

We choose to discuss this case because of its mathematical tractability. Let the transmitting pulse $g(t)$ be a rectangular pulse as

$$g(t) = \frac{1}{\sqrt{T}} \qquad 0 \leq t \leq T, \qquad (23)$$

the equivalent channel filter response, $f(t)$, is then $f(t) = g(t) * c(t)$. Now, at sampling instants T, $2T$, ..., we define $f_k = f((k+1)T), \quad k = 0,1,2,\cdots$.

After normalization, the final equivalent channel response becomes

$$f_k = a^k\sqrt{1-a^2}, \qquad k = 0,1,2,\cdots, \qquad (24)$$

where $a = e^{-T/RC}$.

Considering MSE equalization, we find

$$W'(w) = \frac{\sqrt{1-a^2}\,(1-a\cdot e^{-jw})}{(1-a^2)+\frac{N_0}{2}(1+a^2)-aN_0\cos w} \qquad (25)$$

,

$$Q'(w) = \frac{1-a^2}{(1-a^2)+\frac{N_0}{2}(1+a^2)-aN_0\cos w}, \qquad (26)$$

and

$$q_k = \frac{1-a^2}{\sqrt{A^2-a^2N_0^2}}\left(\frac{A-\sqrt{A^2-a^2N_0^2}}{aN_0}\right)^k \qquad (27)$$

with $A = (1-a^2)+\frac{N_0}{2}(1+a^2)$.

We can also easily find

$$\sum_{k\neq 0} q_k = \frac{1-a^2}{\sqrt{A^2-a^2N_0^2}}\left[\frac{aN_0}{A-\sqrt{A^2-a^2N_0^2}}-1\right]^{-1}. \qquad (28)$$

Notice that, when the received SNR is large (N_0 small), we have $q_0 \cong 1$ and $q_k \cong 0$ for $k \neq 0$.

Further,

$$\|\mathbf{w}_o\|^2 = \frac{(1+a^2)+\frac{N_0}{2}(1-a^2)}{(1-a^2)\left[1+\frac{N_0(1+a^2)}{1-a^2}+\frac{N_0^2}{4}\right]^{3/2}}. \qquad (29)$$

When N_0 is small,

$$\|\mathbf{w}_o\|^2 \cong \frac{1+a^2}{1-a^2}. \qquad (30)$$

Notice that $\|\mathbf{w}_o\|^2 > 1$ as expected.

When using a zero-forcing equalizer, we simply replace 0 for N_0 in the equations from (25) through (29), and (30) becomes exact, then, of course, we also have exactly $q_0 = 1$ and $q_k = 0$ for $k \neq 0$.

Now, substituting (30) into (19), and substituting (30) (with equal sign replacing the approximate equal sign) into (15), we obtain respectively the decision error probabilities for MSE and zero-forcing equalizer for our special example.

Using the 3-dB bandwidth $B = 1/2pRC$ and a symbol period $T = 1/2B = pRC$, we have $a = e^{-T/RC} = e^{-p}$. For large SNR, both MSE and zero-forcing equalizers will have the energy in the equalizer tap weights given by

$$\| \mathbf{w}_o \|^2 = \frac{1+e^{-2p}}{1-e^{-2p}} = 1.0038 \,. \tag{31}$$

Monte Carlo simulation results of decision error probabilities for both MSE and zero-forcing equalizations are presented in Fig. 1 along with the theoretical decision error probability given by (15). In the simulations, we have used a finite equalizer length of 11 for the MSE equalizer with a peak cursor at the center tap. From the figure, it is seen that the simulation results are in excellent agreement both (15).

5 A TELEPHONE CHANNEL

We now consider a typical telephone channel with the discrete-time characteristic given in Fig. 2 (after PROAKIS [8]). Using an equalizer length of 105, we have found the energy in the equalizer tap weights for zero-forcing equalization to be $\| \mathbf{w}_o \|^2 = 1.92$. For MSE equalization, the equalizer tap weight energy is approximately the same when we consider large SNR values.

Computer simulation results of decision error probabilities for both MSE and zero-forcing equalizations along with theoretical decision error probability are presented in Fig. 3. The theoretical decision error probability of (15) is obtained by first finding the norm $\| \mathbf{w}_o \|$ from simulations. It is to be noted that, in this case, at low SNR (below about 20dB), the theoretical error probability slightly deviates from the actual as can be seen in Fig. 3. This is because when SNR is small, the conditions $q_0 \cong 1$, $q_k \cong 0$ for $k \neq 0$ do not exactly hold. Thus (19) becomes less accurate. Computer simulations also show that, at small SNR, the norm $\| \mathbf{w}_o \|$ is less than unity. The overall effect is to have the error probabilities slightly better than the theoretical.

6 CONCLUSIONS

The generalized expression of decision error probability for any linear equalizer in bandlimited channels employing M-ary PAM transmission is in the form of a sum of Q functions. When the SNR is large and the equalizer is in optimum condition, the decision error probability takes a form similar to that of M-ary PAM in an infinite bandwidth channel without ISI with the exception that the channel noise term $N_0/2$ is replaced by $\| \mathbf{w}_o \|^2 \, N_0/2$. In other words, the additive channel noise is enhanced by a factor equal to the energy in the equalizer tap weights. It is this noise enhancement that degrades the equalizer performance. Thus, without channel noise, the ISI can be completely eliminated.

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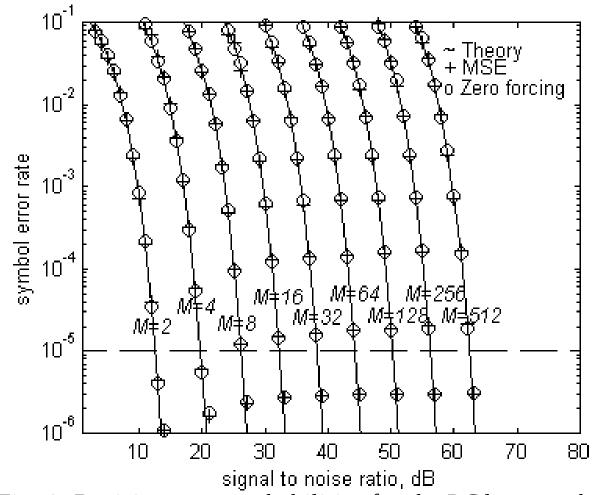

Fig. 1 Decision error probabilities for the *RC* lowpass channel

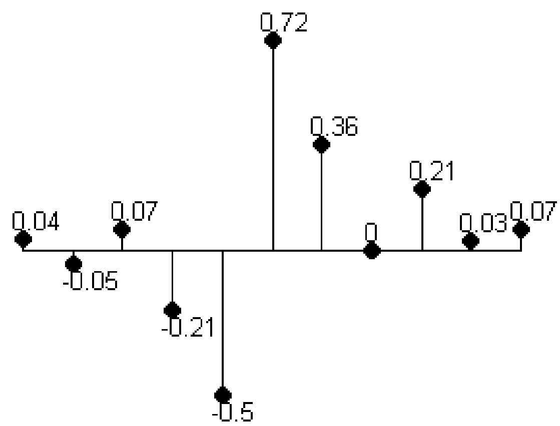

Fig. 2 A discrete channel impulse response

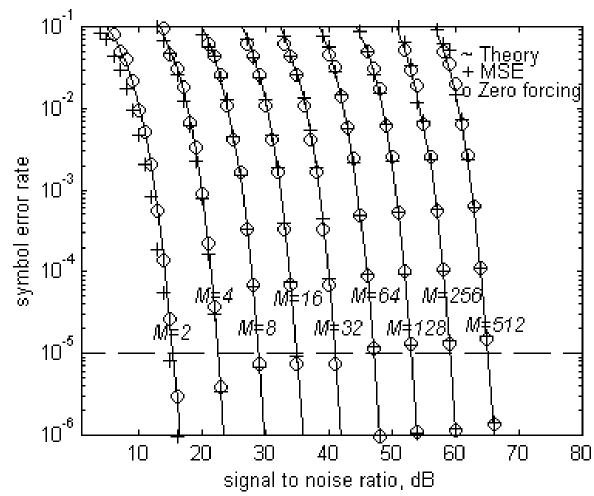

Fig. 3 Decision error probabilities for the realistic channel of Fig. 2