ESTIMATION OF THE MIXING MATRIX FOR UNDERDETERMINED BLIND SOURCE SEPARATION USING SPECTRAL ESTIMATION TECHNIQUES

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ABSTRACT

Blind source separation is concerned with estimating n source signals from m measurements that are generated through an unknown mixing process. In the underdetermined linear case, where the number of measurements is smaller than the number of sources, the solution can be obtained in three stages: represent the signals in a sparse domain, estimate the mixing matrix, and evaluate the sources using the available previous knowledge. This paper deals with the second stage, that can be formulated as to find the peaks location of a probability density function (PDF). It is shown that when the premise of sparse signals is satisfied, the densities resemble the power spectral density (PSD) of sinusoids in noise. The analogy between a PDF and a PSD allows us to apply spectral estimation techniques to determine the mixing matrix. According to the shape of the PDF's, parametric methods for line spectra have been applied.

1 INTRODUCTION

The blind source separation problem consists of estimating n statistically independent sources from m measurements that are an unknown function of the sources. The noise-free linear model for each sample is

$$\mathbf{A}\mathbf{s} = \mathbf{x}, \tag{1}$$

where $\mathbf{s} \in \mathbb{R}^n$ is the source random vector, $\mathbf{x} \in \mathbb{R}^m$ is the measurement random vector, and $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the unknown mixing matrix.

In the underdetermined case, when less measurements than sources are available ($m < n$), the separation process can be divided in three stages [1, 2]: to represent the signals in an appropiate sparse domain, to estimate the mixing matrix, and to invert the underdetermined linear problem (1). In this paper we focus on the second stage.

There have been different approaches taken towards estimating the mixing matrix. Lin et. al use competitive learning in a feature extraction framework [3]. Bofill and Zibulevsky employ a potential function based clustering approach [1]. Wu uses an eigenspread estimation to decide when only one source is active, and uses this information to find the columns of the mixing matrix [4]. On a previous paper [5], we addressed the underdetermined problem with two measurements and reduced the problem of estimating the mixing matrix to estimate the peaks of the probability density function (PDF) of the angles of the measurements. To that end, we used a non-parametric maximum-likelihood approach based on Parzen windowing.

In this paper, we exploit the parallelism between the probability density function (PDF) of a random variable and the power spectral density (PSD) of a related random process [6, 7]. Once the problem is formulated in this way, any spectral estimation technique can be applied to estimate the mixing matrix. In particular, we show that the source sparsity condition leads to a model of sinusoids in noise, so high-resolution parametric methods for line spectra seem the most appropiate.

The paper is organized as follows. In section 2 we introduce a probabilistic sparsity model for the sources and formulate the problem of estimating the mixing matrix as the problem of estimating a PDF. In section 3 we point out the parallelism that exists between estimating the PDF of a discrete random variable and estimating the PSD of a related moment-generating sequence. In section 4 we show that the model of sinusoids in noise is appropriate to estimate the PDF's under the sparsity premise. In section 5 we present numerical results obtained in estimating the mixing matrix of an underdetermined problem with the Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) method [8, 9]. In section 6 we present the conclusions of the work.

2 MIXING MATRIX ESTIMATION

The performance of the solution of the underdetermined BSS problem depends highly on the sparsity of the sources [2, 5]. The higher the probability of the sources of being zero or negligibly small, the better we can do in estimating the sources from the measurements. When the original sources do not satisfy the sparsity condition, a suitable linear transformation should be applied beforehand [1, 10]. To parametrically model sources with

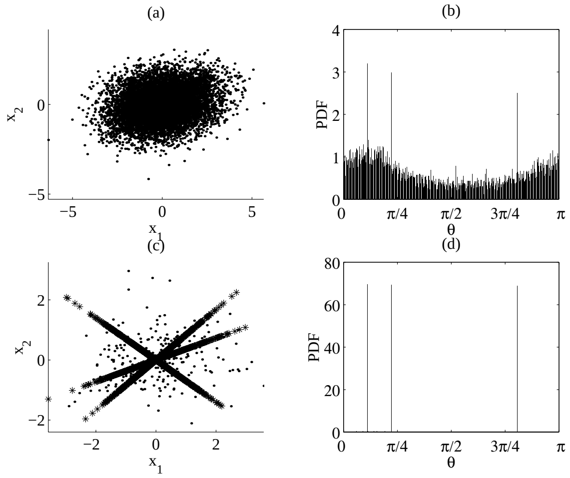

Figure 1: Scatter plot of measurements and histogram of angles for sparsity factors 0.1 —subfigures (a) and (b)— and 0.9 —subfigures (c) and (d).

different degrees of sparsity, the following model for the source densities is used

$$p_{S_j}(s_j) = p_j\,\delta(s_j) + (1 - p_j) f_{S_j}(s_j), \quad j = 1, \ldots, n, \quad (2)$$

where s_j is the j-th source, p_j is the sparsity factor for s_j, and $f_{S_j}(s_j)$ is the PDF when the source j —that is assumed to be zero-mean— is active. This model provides a framework to characterize the behaviour of the algorithms for the underdetermined BSS problem as a function of the sparsity of the sources.

Equation (1) can be interpreted from a geometrical point of view as the projection of the source vectors $\mathbf{s}$ from $\mathbb{R}^n$ into the vector space $\mathbb{R}^m$ of the measurement vectors $\mathbf{x}$. If we denote by $\mathbf{a}_j$ the j-th column of the mixing matrix, so that $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n]$, (1) can be rewritten as

$$\mathbf{x} = \sum_{j=1}^{n} s_j \mathbf{a}_j, \quad (3)$$

that explicitly shows that the measurement vector is a linear combination of the columns of the mixing matrix. According to this interpretation, if at a given time only the j-th source is non zero, the meassurement vector will be collinear with $\mathbf{a}_j$. To illustrate the dependence of the measurements with the sparsity factor of the sources, a case with $m = 2$ and $n = 3$ is shown in figure 1. Subfigures (a) and (c) show the pattern of the vector $\mathbf{x}$ in the measurement space for sources with sparsity factors of 0.1 and 0.9 respectively. It can be observed that for higher sparsity factors, the measurements tend to concentrate more around the three lines that correspond to the columns of the mixing matrix. It is clear from that figures that it should be easier to determine the mixing matrix for sources with a high sparsity factor.

When the number of measurements is two, the columns of the mixing matrix can be parametrized as $\mathbf{a}_j = [\cos(\theta_j), \sin(\theta_j)]^T$. In doing so, we impose that all the columns are of unit norm, that is consistent with the indeterminacy on the scale factors of the sources in the BSS problem [10]. According to this, to determine $\mathbf{A}$ is enough to estimate the angles θ_j, $j = 1, \ldots, m$. In principle, the angles θ_j could have any value in the range $[0, 2\pi]$, but since the BSS problem also exhibits a sign indeterminacy in the sources (the separation procedure does not distinguish between s_j and $-s_j$), the angles θ_j and $\theta_j + \pi$ would be indistinguishable. Therefore, the angles will be considered to belong to the interval $[0, \pi]$.

A first approach to estimate the angles that characterize the mixing matrix from the measurements could be to evaluate the angle of $\mathbf{x}$ in the measurement space as

$$\theta = \arctan \frac{x_2}{x_1} \mod \pi,$$

where x_1 and x_2 are the components of $\mathbf{x}$, and represent an histogram of the values. The zero measurements are simply omitted, as they have no well-defined angle. Figures 1(b) and 1(d) show histograms with 360 bins for sparsity factors of 0.1 and 0.9 respectively. It can be observed that even for an sparsity factor as low as 0.1 —only ten percent of the sources are negligible— the peaks of the PDF are clearly identifiable [5]. The angle resolution that can be obtained with the histogram highly depends on the number of bins used. If we increase the number of bins to improve the resolution, the number of samples on each bin will decrease and the variance of the estimation will be higher.

If one looks at the histograms of figures 1(b) and 1(d), they soon reveal themself as reminiscent of the power spectral density of sinusoids in noise. With that thought in mind, a promising method could be to transform the PDF estimation problem in a spectral estimation one, as we show in the next section, and to apply high-resolution techniques to identify the angles of the mixing matrix.

3 PDF AS A PSD

Let U be a continuous random variable in the finite range $[U_{\min}, U_{\max}]$, and $u[n]$, $n = 0, \ldots, N-1$, a sequence consisting of N independent realizations of U. If the linear transformation

$$\Omega = -\pi + \frac{2\pi}{U_{\max} - U_{\min}}(U - U_{\min}) \quad (4)$$

is applied, Ω is a new continuous random variable in the range $[-\pi, \pi]$. The sequence of N realizations $\omega[n]$ is obtained from $u[n]$ applying the same linear transformation. We denote by $\Phi_\Omega(\omega)$ the PDF of Ω, that is zero outside of the interval $[-\pi, \pi]$, and build $\tilde{\Phi}_\Omega(\omega)$ as the periodic extension of $\Phi_\Omega(\omega)$ with period 2π. According to this periodicity, $\tilde{\Phi}_\Omega(\omega)$ can be considered as the Fourier transform of a certain sequence,

$$\tilde{\Phi}_\Omega(\omega) = \mathcal{F}\{\phi_\Omega[k]\}. \quad (5)$$

This is equation shows the way to estimate PDF's using spectral estimation techniques [6, 7]: if we are able to determine the sequence $\phi_\Omega[k]$, its PSD evaluated in the interval $[-\pi, \pi]$ will be the PDF of Ω. Next we show how to find $\phi_\Omega[k]$. According to (5), the sequence $\phi_\Omega[k]$ is the inverse Fourier transform of $\tilde{\Phi}_\Omega(\omega)$

$$\phi_\Omega[k] = \frac{1}{2\pi} \int_0^{2\pi} \Phi_\Omega(\omega) e^{j\omega k}\, d\omega, \quad k = 0, 1, \ldots, \tag{6}$$

where $\Phi_\Omega(\omega)$ is used instead of $\tilde{\Phi}_\Omega(\omega)$ since both coincide on the integration interval. Since $\Phi_\Omega(\omega)$ is the PDF of Ω, the integral in (6) can be considered as the mean value of the exponential that is multiplying to the PDF

$$\phi_\Omega[k] = \frac{1}{2\pi} E\{\exp(j\Omega k)\}, \quad k = 0, 1, \ldots;$$

that is, the sequence $\phi_\Omega[k]$ consist of scaled samples of the moment generating function of the random variable Ω. Therefore, to estimate the PDF of Ω, the sequence $\phi_\Omega[k]$ has to be estimated before. To that end, the sample moment estimator

$$\hat{\phi}_\Omega[k] = \frac{1}{2\pi N} \sum_{n=0}^{N-1} \exp\left(j\omega[n]k\right), \quad k = 0, 1, \ldots, \tag{7}$$

can be used. According to the preceding discussion, the procedure to estimate the PDF of U from the sequence $u[n]$ can be summarized as follows:

1. Generate $\omega[n]$ applying (4) to $u[n]$.
2. Generate $\hat{\phi}_\Omega[k]$ using (7).
3. Use (5) to estimate the PDF $\Phi_\Omega(\omega)$ applying a spectral estimation technique.
4. Invert the linear transformation (4) to obtain the PDF of the original random variable U.

4 SIGNAL MODEL

In our problem, the role of U is played by the random variable Θ, whose realizations are the angles $\theta[n]$ of the measurement vector $\mathbf{x}[n]$. Since these angles are already periodic with a period of 2π, the linear transformation from U to Ω and the periodic extension is not strictilly necesary in this case. However, because of the sign ambiguity in the BSS problem, Θ is considered as a random variable on the range $[0, \pi]$.

By observing the histograms of figure 1, it is apparent that they resemble quite close the PSD of sinusoids in noise. Therefore, from the whole set of spectral estimation methods, the most appropriate for the estimation of the angles of the mixing matrix seem to be those oriented to the estimation of line spectra. We would like to point out two important aspects about the signal model by looking carefully at figures 1(b) and 1(d) considered as PSD's. On the one hand, it is clear than n spectral peaks —one for each column of the mixing matrix— are present, so the model order should be chosen equal to the number of sources. On the other hand, it is also quite obvious than there is a noise that depends on the sparsity factor of the sources. It should be stressed than the cause of this noise is the simultaneity of the sources —recall that, according to (3), the measuraments are collinear with the columns of the mixing matrix when only one source is active—, and has nothing to do with noise in the measurements, since the model of (1) is noise-free. Even if this noise does not adhere to the circular white noise model that is usually supposed in the parametric methods for line spectra, very accurate frequency estimations are obtained with subspace-based methods, as we will show in the numerical results. With those considerations, the PSD that we are trying to estimate conforms with the following model

$$\Phi_\Omega(\omega) = 2\pi \sum_{j=1}^{n} \alpha_j^2 \delta(\omega - \omega_j) + \sigma^2,$$

where α_j are the sinusoids amplitudes, $\delta(\omega)$ is the Dirac impulse, and σ^2 is the noise level. The only parameters to estimate are the frequencies ω_j, $j = 1, \ldots, n$; from which the angles of the mixing matrix are readly obtained.

5 NUMERICAL RESULTS

An underdetermined problem with $m = 2$ measurements and $n = 3$ sources has been studied. The sources realizations have been generated according to model in (2), with $f_{S_j}(s_j)$ as Gaussian densities of zero mean and unit variance. Two kind of Montecarlo simulations have been performed. On the one hand we have characterized the performance of ESPRIT in estimating the mixing matrix for sources with different sparsity factors. On the other hand we have evaluated the performance against the number of available measurement realizations. The figure of merit we use to characterize the performance of the estimation procedures is the mean-squared error (MSE) in angle estimation, defined as

$$\text{MSE} = \frac{1}{Q} \sum_{q=1}^{Q} \frac{1}{m} \sum_{j=1}^{m} (\theta_j - \hat{\theta}_j)^2,$$

where Q is the number of simulations used in the Montecarlo method, m is the number of sources, θ_j are the true angles, and $\hat{\theta}_j$ are the estimations.

In figure 2 we show the MSE obtained when estimating the mixing matrix both with the histogram (using 360 bins) and with ESPRIT. The error is plotted against the sparsity factor of the sources. One thousand realizations of the measurements have been simulated, and one thousand values of $\hat{\phi}_\Omega[k]$ have been generated according to (7). The results shown have been obtained by a Montecarlo simulation with twenty mixing matrices. As it was expected, the error is greater for lower

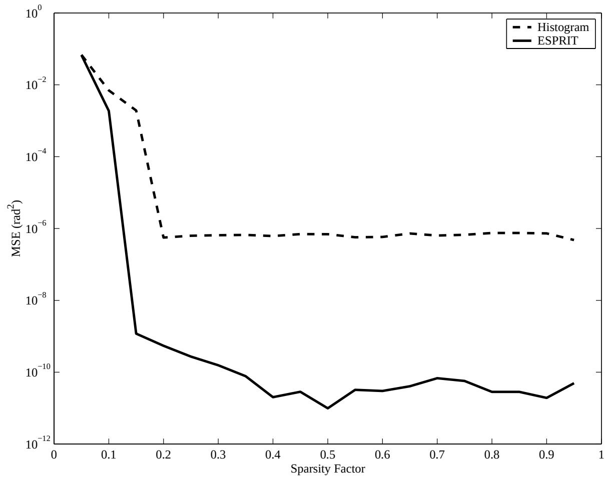

Figure 2: MSE in angle estimation versus sparsity factor for ESPRIT (solid line) and histogram (dashed line) with 360 bins.

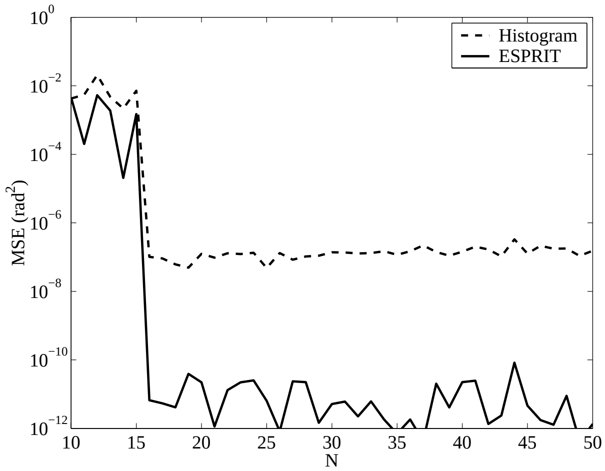

Figure 3: MSE versus the number of available realizations for ESPRIT (solid line) and histogram (dashed line) with 360 bins. The sparsity factor is 0.5.

sparsity factors. For sparsity factors bigger than 0.2, a big portion of the MSE in the histogram estimation is due to the finite bin length. It can be observed than very accurate estimations (MSE on the order of 10^{-10}) are achieved with ESPRIT. In figure 3 we show the MSE against the number of available realizations of the measurements for a fixed sparsity factor of 0.5. It can be observed that, for this sparsity factor, there is a significant point at about fifteen available realizations: the results greatly improve when the number of realizations is slightly greater. The results have been obtained from a Montecarlo simulation with fifty mixing matrices.

6 CONCLUSIONS

We have formulated the problem of finding the mixing matrix in the instantaneous underdetermined linear BSS as estimating the peaks of a PDF. To estimate those peaks we have exploited a duality between the PDF of a random variable and the PSD of a derived random process that has allowed us to apply spectral estimation techniques in the estimation of the mixing matrix. Since the original PDF resembles the PSD of sinusoids in noise, high-resolution parametric methods for line spectra are the most appropriate. We have studied the performance of ESPRIT in estimating the mixing matrix as a function of the sparsity factor of the sources, and conclude that it provides very good results even for very low sparsity factors. We have also characterized the performance of the estimation versus the number of realizations available. As future lines of work, we have identified to tracking of the mixing matrix in a nonstationary environment, using non-linear least-squares techniques to deal with the colored noise, and applying multidimensional spectral estimation techniques for cases with more that two measurements.

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