HYPERCOMPLEX OPERATORS AND VECTOR CORRELATION

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ABSTRACT

Auto-, cross- and phase-correlation are standard techniques in image and signal processing. Recently published hypercomplex forms have been applied to vector images using quaternion Fourier transforms. The various permutations of ordering and conjugation of the basic definition are compounded by realization of a closed set of hypercomplex operators leading to multiple derivations and the possibility of unintentional *mirror* responses. This paper identifies the necessity of carefully choosing the form of the correlation function, presents for the first time the set of equivalent forms for hypercomplex correlation and demonstrates how these can be validated using hypercomplex operator algebra.
Indexing: image/signal processing, vector/hypercomplex correlation, hypercomplex/quaternion operators.

1 INTRODUCTION

The discrete derivation of vector auto- and cross-correlation functions in the spatial-frequency domain using a hypercomplex generalization of the Wiener-Khintchine theorem was presented in [1]:

$$r(m,n) = \sum_{q=0}^{M-1}\sum_{p=0}^{N-1} f(q,p)\overline{g(q-m,p-n)}$$
$$= \mathcal{F}^{-T}\left\{F^{+T}[\boldsymbol{u}]\,\overline{G_{\parallel}^{+T}[\boldsymbol{u}]}\right\} + \mathcal{F}^{+T}\left\{F^{+T}[\boldsymbol{u}]\,\overline{G_{\perp}^{+T}[\boldsymbol{u}]}\right\} \quad (1)$$

where $\boldsymbol{u} = (u,v)$. The other symbols are explained later. In [2], an alternative form of hypercomplex cross-correlation, based on a continuous definition, was derived:

$$f \star g(\boldsymbol{x_a}) = \int_{-\infty}^{\infty} \overline{f(\boldsymbol{x})} g(\boldsymbol{x}+\boldsymbol{x_a})d\boldsymbol{x}$$
$$= \mathcal{F}^{\mp T}\left\{\overline{F^{\pm 1}[\boldsymbol{u}]}G_{\parallel}^{\pm T}[\boldsymbol{u}] + \overline{F^{\mp 1}[\boldsymbol{u}]}G_{\perp}^{\pm T}[\boldsymbol{u}]\right\} \quad (2)$$

where $\boldsymbol{x} = (x,y)$ and $\boldsymbol{x_a}$ is the shifted vector component of the correlation. While these two forms yield numerically equivalent results, the registration of the reference and object images indicates a reversal. The two, reportedly equivalent basic correlation definitions [3](pp. 90), effectively produce a function reversal as shown in Figure 1 not normally apparent when processing real data. These spectra have been produced by the cross-correlation of two images of the same scene, spatially shifted from each other by some known amount. An unshifted image, or auto-correlation, would yield a peak (white dot) in the center of the image. The mirrored results, demonstrated in Figure 1, are not due to the different methods of derivation. However, the different versions of the hypercomplex form add to complexity of validating the desired result since they incorporate symmetrical relationships which are inherent in quaternionic functions.

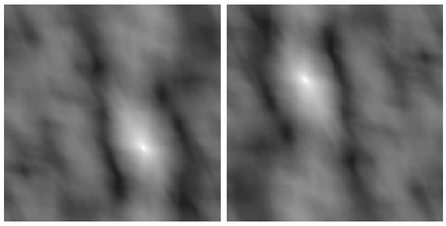

Figure 1: Modulus responses for the form in Eqn (1) –left and Eqn (2) –right.

This paper begins with a review in § 2 of the basic definitions of quaternion Fourier transforms and operators used in the derivation of hypercomplex formula in the spatial-frequency domain. This is followed, in § 3, by a review of the basic definition of correlation and relates it to the multidimensional problem space. § 4 demonstrates a previously unpublished derivation of one of the hypercomplex correlation forms. Finally § 5 lists the equivalent forms of both derivation methods and demonstrates the use of the hypercomplex operator graph in the algebraic validation of equivalence.

2 HYPERCOMPLEX OPERATORS

Quaternions (also referred to as *hypercomplex numbers*) are an extension of complex numbers to four dimensions, originally proposed by Hamilton in 1843. They can be considered as a complex number with a vector imaginary part consisting of three mutually orthogonal components. Given a quaternion $q = w + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ its quaternion conjugate is $\bar{q} = w - x\,\boldsymbol{i} - y\,\boldsymbol{j} - z\,\boldsymbol{k}$. A *pure* quaternion has a zero real part ($w = 0$) and a *unit* quaternion has a unit modulus. It is often useful to consider a quaternion as composed of a Scalar and a Vector part, represented by $q = S(q) + \boldsymbol{V}(q)$, where

$\boldsymbol{V}(q) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$.

An RGB color image may be represented using quaternions by encoding the red, green and blue channels of the image as a pure quaternion such that the image function is given by, $f(x,y) = r(x,y)\,\boldsymbol{i} + g(x,y)\,\boldsymbol{j} + b(x,y)\,\boldsymbol{k}$.

In order to simplify both the notation and explain rotations in color-space it is easier to consider the polar form of a quaternion. Euler's formula for the complex exponential generalizes to the hypercomplex form: $e^{\boldsymbol{\mu}\Phi} = \cos\Phi + \boldsymbol{\mu}\sin\Phi$ where $\boldsymbol{\mu}$ is a pure unit quaternion. Any quaternion may be represented in polar form by, $q = |q|\ e^{\boldsymbol{\mu}\Phi}$ where $\boldsymbol{\mu}$ and Φ are referred to as the *eigenaxis* and the *eigenangle* respectively, We generally refer to the former simply as the *axis* and the latter as the *phase*. The eigenaxis, or axis, is computed as $\boldsymbol{\mu} = \boldsymbol{V}(q)/\left|\boldsymbol{V}(q)\right|$ with the only exception being when $\boldsymbol{V}(q) = 0$, in which case $\boldsymbol{\mu}$ is undefined. The eigenangle, or phase, is computed as $\Phi = \tan^{-1}\left|\boldsymbol{V}(q)\right|/S(q)$ and is always positive in the range $0 \le \Phi \le \pi$.

The hypercomplex Fourier transform pairs are given by:

$$\mathcal{F}^{\pm 1}\left[f(\boldsymbol{x})\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mp\boldsymbol{\mu}\boldsymbol{u}^T\boldsymbol{x}} f(\boldsymbol{x})d\boldsymbol{x} = F^{\pm 1}\left[\boldsymbol{u}\right] \Leftrightarrow$$

$$\mathcal{F}^{\mp 1}\left[F^{\pm 1}\left[\boldsymbol{u}\right]\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm\boldsymbol{\mu}\boldsymbol{u}^T\boldsymbol{x}} F^{\pm 1}\left[\boldsymbol{u}\right]d\boldsymbol{u} = f(\boldsymbol{x}) \quad (3)$$

$$\mathcal{F}^{\pm T}\left[f(\boldsymbol{x})\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\boldsymbol{x}) e^{\mp\boldsymbol{\mu}\boldsymbol{u}^T\boldsymbol{x}} d\boldsymbol{x} = F^{\pm T}\left[\boldsymbol{u}\right] \Leftrightarrow$$

$$\mathcal{F}^{\mp T}\left[F^{\pm T}\left[\boldsymbol{u}\right]\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^{\pm T}\left[\boldsymbol{u}\right]\, e^{\pm\boldsymbol{\mu}\boldsymbol{u}^T\boldsymbol{x}} d\boldsymbol{u} = f(\boldsymbol{x}) \quad (4)$$

where $\boldsymbol{x} = (x, y)$, $\boldsymbol{u} = (u, v)$ and $\boldsymbol{\mu}$ is an arbitrary unit vector which defines the axis of the transformation. The latter pair are called *transpose* transforms. The transforms of spatially shifted functions are given by the following, using the substitution $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{x_a}$, $\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x_a}$ and two directional variables $\alpha = \pm 1$ and $\beta = \pm 1$:

$$\mathcal{F}^{\alpha}\left[f(\boldsymbol{x} + \beta\boldsymbol{x_a})\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\boldsymbol{\mu}\boldsymbol{u}^T\boldsymbol{x}} f(\boldsymbol{x} + \beta\boldsymbol{x_a})d\boldsymbol{x} \Rightarrow$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\boldsymbol{\mu}\boldsymbol{u}^T(\boldsymbol{x}' - \beta\boldsymbol{x_a})} f(\boldsymbol{x}')d\boldsymbol{x}' = e^{\alpha\beta\boldsymbol{\mu}\boldsymbol{u}^T\boldsymbol{x_a}}\mathcal{F}^{\alpha}\left[f(\boldsymbol{x})\right] \quad (5)$$

A similar argument applies to the transpose transform pair yielding an exponential term on the right hand side. Note that there is an inverse relationship here between the sign of the transform kernel and a negative correlation shift component which does not apply to a positive shift.

Each of the transform pairs can be decomposed into parallel and perpendicular components. Given a *pure* quaternion $\boldsymbol{u}$, and a second pure *unit* quaternion $\boldsymbol{v}$, $\boldsymbol{u}$ may be decomposed into components parallel and perpendicular to $\boldsymbol{v}$ using: $\boldsymbol{u}_\perp = (\boldsymbol{u} + \boldsymbol{v}\boldsymbol{u}\boldsymbol{v})/2$ where $\boldsymbol{v} \perp \boldsymbol{u}_\perp$ and $\boldsymbol{u}_\parallel = (\boldsymbol{u} - \boldsymbol{v}\boldsymbol{u}\boldsymbol{v})/2$ with $\boldsymbol{v} \parallel \boldsymbol{u}_\parallel$ [1]. This follows from Coxeter [4] where: $\boldsymbol{u} = \boldsymbol{v}\boldsymbol{u}\boldsymbol{v}$, $\forall\boldsymbol{u} \perp \boldsymbol{v}$ and $\boldsymbol{u} = -\boldsymbol{v}\boldsymbol{u}\boldsymbol{v}$, $\forall\boldsymbol{u} \parallel \boldsymbol{v}$. Geometrically, $\boldsymbol{v}\boldsymbol{u}\boldsymbol{v}$ represents a reflection of $\boldsymbol{u}$ in the plane normal to $\boldsymbol{v}$. Writing $\boldsymbol{u} = \boldsymbol{u}_\perp + \boldsymbol{u}_\parallel$ we have:

$$\boldsymbol{v}\boldsymbol{u}\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{u}_\perp + \boldsymbol{u}_\parallel)\boldsymbol{v} = \boldsymbol{v}\boldsymbol{u}_\perp\boldsymbol{v} + \boldsymbol{v}\boldsymbol{u}_\parallel\boldsymbol{v} = \boldsymbol{u}_\perp - \boldsymbol{u}_\parallel$$

Therefore: $\boldsymbol{u} + \boldsymbol{v}\boldsymbol{u}\boldsymbol{v} = (\boldsymbol{u}_\perp + \boldsymbol{u}_\parallel) + (\boldsymbol{u}_\perp - \boldsymbol{u}_\parallel) = 2\boldsymbol{u}_\perp$ from which $\boldsymbol{v} \perp \boldsymbol{u}_\perp$ follows and similarly for $\boldsymbol{u} - \boldsymbol{v}\boldsymbol{u}\boldsymbol{v}$ from which $\boldsymbol{v} \parallel \boldsymbol{u}_\parallel$ follows. It can be shown that, if a *full* quaternion is decomposed about a vector $\boldsymbol{v}$, that, $q_\parallel = S(q) + \boldsymbol{V}_\parallel(q)$ and $q_\perp = \boldsymbol{V}_\perp(q)$. Note that the scalar component of q is part of the parallel component. This is somewhat counter-intuitive. Parallel quaternions (strictly co-planar quaternions, or quaternions with parallel vector parts) commute. If q is a full quaternion and $\boldsymbol{p}$ is a vector where $q \perp \boldsymbol{p}$, we can reorder them such that $q\boldsymbol{p} = \boldsymbol{p}\overline{q}$. The proof for this is given by:

$$q\boldsymbol{p} = [S(q) + \boldsymbol{V}(q)]\boldsymbol{p} = S(q)\boldsymbol{p} + \boldsymbol{V}(q)\boldsymbol{p} \quad \text{but}$$
$$\boldsymbol{p}\overline{q} = \boldsymbol{p}[S(q) - \boldsymbol{V}(q)] = S(q)\boldsymbol{p} - \boldsymbol{p}\boldsymbol{V}(q)$$

Two perpendicular vectors (here $\boldsymbol{p}$ and $\boldsymbol{V}(q)$) reverse their sign on reordering since the product of any two vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ is given by $\boldsymbol{v}\boldsymbol{u} = -\boldsymbol{v}\cdot\boldsymbol{u} + \boldsymbol{v}\times\boldsymbol{u}$, and when $\boldsymbol{v} \perp \boldsymbol{u}$ the dot product is zero.

Decomposing the transform pairs about the axis $\boldsymbol{\mu}$:

$$\urcorner_{\boldsymbol{\mu}}\lceil F^{\pm 1}\left[\boldsymbol{u}\right]\rfloor = F_\parallel^{\pm 1}\left[\boldsymbol{u}\right] + F_\perp^{\pm 1}\left[\boldsymbol{u}\right]$$
$$\urcorner_{\boldsymbol{\mu}}\lceil F^{\pm T}\left[\boldsymbol{u}\right]\rfloor = F_\parallel^{\pm T}\left[\boldsymbol{u}\right] + F_\perp^{\pm T}\left[\boldsymbol{u}\right] \quad (6)$$

where $\urcorner_{\boldsymbol{\mu}}\lceil\ \ \rfloor$ denotes the decomposition function about the axis $\boldsymbol{\mu}$. The result is referred to as the *symplectic* form.

The relationship between the symplectic components and the four transform pairs is fundamental to the idea of hypercomplex operator identities. By decomposing the object function $f(x)$ into symplectic form it can be shown that while the parallel components can be exchanged directly between transform kernels, the perpendicular components can only be exchanged for the transposed kernel in the reverse direction. This leads to the identities:

$$F_\parallel^{\pm 1}\left[\boldsymbol{u}\right] = F_\parallel^{\pm T}\left[\boldsymbol{u}\right] \quad F_\perp^{\pm 1}\left[\boldsymbol{u}\right] = F_\perp^{\mp T}\left[\boldsymbol{u}\right] \quad (7)$$

The product of a transform pair yields an identity operator I and an operator relationship between the transform, conjugate and image reversal such that:

$$\overline{\mathcal{F}^{+1}\left[f(\boldsymbol{x})\right]} = \mathcal{F}^{+T}\left[\overline{f(-\boldsymbol{x})}\right]$$
$$\overline{\mathcal{F}^{+T}\left[f(\boldsymbol{x})\right]} = \mathcal{F}^{+1}\left[\overline{f(-\boldsymbol{x})}\right] \quad (8)$$

This relationship can be rewritten in operator notation as:

$$K\mathcal{F}^{+1} = \mathcal{F}^{+T}KR \quad \text{and} \quad K\mathcal{F}^{+T} = \mathcal{F}^{+1}KR \quad (9)$$

where K is the conjugate operator and R is image reversal. Given that $q_1\,q_2 = \overline{\overline{q_2}\,\overline{q_1}}$ a further set of operator identities can be derived from Eqn (9):

$$\mathcal{F}^{+T} = K\mathcal{F}^{-1}K \quad \mathcal{F}^{+1} = K\mathcal{F}^{-T}K$$
$$I = K\mathcal{F}^{+T}K\mathcal{F}^{+1} \quad I = K\mathcal{F}^{+1}K\mathcal{F}^{+T} \quad (10)$$

From Eqns (9) and (10) it can be shown that: $\mathcal{F}^{\pm T}\mathcal{F}^{\pm T} = R$ and $\mathcal{F}^{\pm 1}\mathcal{F}^{\pm 1} = R$.

The complete relationship of all of these operators can be demonstrated by an operator graph, as shown in Figure 2.

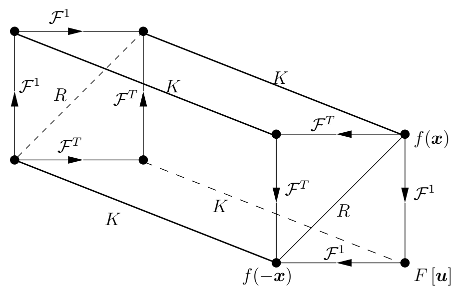

Figure 2: Hypercomplex operator graph.

The directed lines indicate the forward direction of the particular transform. Traversing the line in the opposite direction would yield the reverse transform. The conjugate reversal operators are not directed since $KK = I$ and $RR = I$ and any closed-loop path around the graph results in I.

3 A REVIEW OF CORRELATION

Unlike the convolution of two functions, the correlation of two functions does not commute. Consider:

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau)g(\tau - t)d\tau = \int_{-\infty}^{\infty} f(\tau + t)g(\tau)d\tau \tag{11}$$

where substituting $\tau - t = \tau'$ and $\tau = \tau' + t$ yields the second form. In [5] the convolution of two functions is illustrated as the integral of the product of the two functions with the second reversed. This is distinctly different from the correlation of two functions which does not involve a reversal.

For the correlation of real functions, Eqn (11) is entirely sufficient. However, for complex functions a conjugate operation is required to ensure the relation of the auto-correlation function ($f(t) \star f(t)$) to the power spectrum ($S(\omega) = 1/2T\,|x(\omega)|^2$) as required by the Wiener-Khintchine theorem, see [6] § 2. 2. This effectively ensures that the power spectrum of a complex auto-correlation is real-valued.

The extension to the standard forms of cross-correlation function in Eqn (11) is given in [5, 6]:

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau)\overline{g(\tau - t)}d\tau = \int_{-\infty}^{\infty} \overline{f(\tau)}g(\tau + t)d\tau \tag{12}$$

where the assumption made for the latter is that since $x(f) = \overline{x(-f)}$ and $x(-f) = \overline{x(f)}$ the reversal cancels the conjugate. However, the latter is not always valid since the assumption is not guaranteed. For complex numbers $Z_1\overline{Z_2} \neq \overline{Z_1}Z_2$. When images are represented by complex, or hypercomplex, numbers they are often purely imaginary. In such cases $Z_1\overline{Z_2} = \overline{Z_1}Z_2$ and the assumption is invalid. In general the literature does not give significance to this since the direction of shift and/or the sign of the spectra is immaterial for most correlation problems. However, in the case of image registration and global color correction using vector phase-correlation [2] direction is fundamental.

The equivalent form of Eqn (12) should therefore be:

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau + t)\overline{g(\tau)}d\tau \tag{13}$$

The choice of which function to conjugate is only arbitrary when computing *pure* hypercomplex numbers.

4 DERIVATION OF ALTERNATE FORMS

From the former correlation definition given in Eqn (12) the hypercomplex form can be derived as follows:

$$\mathcal{F}^{\pm T}\left[f \star g(\boldsymbol{x_a})\right] =$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(\boldsymbol{x})\overline{g(\boldsymbol{x} - \boldsymbol{x_a})}d\boldsymbol{x}\right\} e^{\mp\boldsymbol{\mu u}^T\boldsymbol{x}}d\boldsymbol{x} \tag{14}$$

Substituting for $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{x_a}$ and $\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{x_a}$ and rearranging yields:

$$= \int_{-\infty}^{\infty} f(\boldsymbol{x})\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{g(\boldsymbol{x}')}e^{\pm\boldsymbol{\mu u}^T\boldsymbol{x}'}d\boldsymbol{x}'\right\} e^{\mp\boldsymbol{\mu u}^T\boldsymbol{x_a}}d\boldsymbol{x_a}$$

where the negated power on the second exponential term inverts the sign within the brackets.

Incorporate the exponential term into the conjugate operation by reordering and negating.

$$= \int_{-\infty}^{\infty} f(\boldsymbol{x})\overline{\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\mp\boldsymbol{\mu u}^T\boldsymbol{x}'}g(\boldsymbol{x}')d\boldsymbol{x}'\right\}} e^{\mp\boldsymbol{\mu u}^T\boldsymbol{x_a}}d\boldsymbol{x_a}$$

Substitute the bracketed term for a left-hand transform:

$$= \int_{-\infty}^{\infty} f(\boldsymbol{x})\,\overline{G^{\pm 1}[\boldsymbol{u}]}\,e^{\mp\boldsymbol{\mu u}^T\boldsymbol{x_a}}d\boldsymbol{x_a} \tag{15}$$

Decompose the transformed term:

$$= \int_{-\infty}^{\infty} f(\boldsymbol{x})\left\{\overline{G_{\parallel}^{\pm 1}[\boldsymbol{u}]} + \overline{G_{\perp}^{\pm 1}[\boldsymbol{u}]}\right\} e^{\mp\boldsymbol{\mu u}^T\boldsymbol{x_a}}d\boldsymbol{x_a} \tag{16}$$

Since the exponential term and the transform on left are parallel they commute and can be reordered directly. The exponential term and the transform on the right are perpendicular and therefore reordering conjugates the exponential term. Therefore multiplying out the bracket and applying both rules yields:

$$\begin{aligned} &= \int_{-\infty}^{\infty} f(\boldsymbol{x})e^{\mp\boldsymbol{\mu u}^T\boldsymbol{x_a}}d\boldsymbol{x_a}\,\overline{G_{\parallel}^{\pm 1}[\boldsymbol{u}]} \\ &+ \int_{-\infty}^{\infty} f(\boldsymbol{x})e^{\pm\boldsymbol{\mu u}^T\boldsymbol{x_a}}d\boldsymbol{x_a}\,\overline{G_{\perp}^{\pm 1}[\boldsymbol{u}]} \end{aligned} \tag{17}$$

Substituting for a forward transpose transform on the left and a inverse transpose transform on the right:

$$= \left\{F^{+T}[\boldsymbol{u}]\,\overline{G_{\parallel}^{\pm 1}[\boldsymbol{u}]} + F^{-T}[\boldsymbol{u}]\,\overline{G_{\perp}^{\pm 1}[\boldsymbol{u}]}\right\} \tag{18}$$

The difference between this form and the alternative, discrete derivation, form in [1] is the use of a single reverse transform kernel. The form presented here requires the use of mixed standard and transpose transforms whereas the alternative form requires only one type of transform pair. In either of the derived cases a total of four hypercomplex transforms are required to implement the correlation function.

5 COMPARING FORMS

Using the discrete method of derivation in [1] and that given in § 4 a set of equivalent forms for Eqns (12) and (13) can be derived. There are:

$$\mathcal{F}^{+T}\left\{f(\boldsymbol{x}_0)\overline{g(\boldsymbol{x}_0-\boldsymbol{x})}\right\} =$$

$$\mathcal{F}^{-T}\left\{F^{+T}[\boldsymbol{u}]\,\overline{G_\parallel^{+T}[\boldsymbol{u}]}\right\} + \mathcal{F}^{+T}\left\{F^{+T}[\boldsymbol{u}]\,\overline{G_\perp^{+T}[\boldsymbol{u}]}\right\} \quad (19)$$

$$\equiv \mathcal{F}^{\mp T}\left\{F^{+T}[\boldsymbol{u}]\,\overline{G_\parallel^{\pm 1}[\boldsymbol{u}]} + F^{-T}[\boldsymbol{u}]\,\overline{G_\perp^{\pm 1}[\boldsymbol{u}]}\right\} \quad (20)$$

$$\mathcal{F}^{+T}\left\{f(\boldsymbol{x}_0+\boldsymbol{x})\overline{g(\boldsymbol{x}_0)}\right\} =$$

$$\mathcal{F}^{-T}\left\{F^{+T}[\boldsymbol{u}]\,\overline{G_\parallel^{+T}[\boldsymbol{u}]}\right\} + \mathcal{F}^{+T}\left\{F^{+T}[\boldsymbol{u}]\,\overline{G_\perp^{+T}[\boldsymbol{u}]}\right\} \quad (21)$$

$$\equiv \mathcal{F}^{\mp 1}\left\{F_\parallel^{\pm 1}[\boldsymbol{u}]\,\overline{G^{\pm T}[\boldsymbol{u}]} + F_\perp^{\pm 1}[\boldsymbol{u}]\,\overline{G^{\mp T}[\boldsymbol{u}]}\right\} \quad (22)$$

Note that the discrete derivations yields an identical form. These have all been proven experimentally, yielding identical spectra for the same image set.

Using the hypercomplex identities and the operator graph given in Figure 2, the algebraic validation of these equivalent forms can be conducted as follows: From Eqn (19) decompose image F and multiply out the terms.

$$= \mathcal{F}^{-T}\left\{F_\parallel^{+T}[\boldsymbol{u}]\,\overline{G_\parallel^{+T}[\boldsymbol{u}]} + F_\perp^{+T}[\boldsymbol{u}]\,\overline{G_\parallel^{+T}[\boldsymbol{u}]}\right\}$$
$$+\mathcal{F}^{+T}\left\{F_\parallel^{+T}[\boldsymbol{u}]\,\overline{G_\perp^{+T}[\boldsymbol{u}]} + F_\perp^{+T}[\boldsymbol{u}]\,\overline{G_\perp^{+T}[\boldsymbol{u}]}\right\} \quad (23)$$

Repeat for Eqn (20) ignoring the scale factor which is incorporated in the other form.

$$= \mathcal{F}^{\mp T}\left\{F_\parallel^{+T}[\boldsymbol{u}]\,\overline{G_\parallel^{\pm 1}[\boldsymbol{u}]} + F_\perp^{+T}[\boldsymbol{u}]\,\overline{G_\parallel^{\pm 1}[\boldsymbol{u}]}\right\}$$
$$+\mathcal{F}^{\mp T}\left\{F_\parallel^{-T}[\boldsymbol{u}]\,\overline{G_\perp^{\pm 1}[\boldsymbol{u}]} + F_\perp^{-T}[\boldsymbol{u}]\,\overline{G_\perp^{\pm 1}[\boldsymbol{u}]}\right\} \quad (24)$$

From Eqns (23) and (24) it is clear that the first bracketed term is already equivalent. Considering only the second:

$$\mathcal{F}^{+T}\left\{F_\parallel^{+T}[\boldsymbol{u}]\,\overline{G_\perp^{+T}[\boldsymbol{u}]} + F_\perp^{+T}[\boldsymbol{u}]\,\overline{G_\perp^{+T}[\boldsymbol{u}]}\right\}$$
$$\Rightarrow \mathcal{F}^{-T}\left\{F_\parallel^{-T}[\boldsymbol{u}]\,\overline{G_\perp^{+1}[\boldsymbol{u}]} + F_\perp^{-T}[\boldsymbol{u}]\,\overline{G_\perp^{+1}[\boldsymbol{u}]}\right\}$$

Applying the operator identity, $\mathcal{F}^{-T} = K\mathcal{F}^{+1}K$, the latter becomes:

$$= \overline{\mathcal{F}^{+1}\overline{\left\{F_\parallel^{-T}[\boldsymbol{u}]\,\overline{G_\perp^{+1}[\boldsymbol{u}]} + F_\perp^{-T}[\boldsymbol{u}]\,\overline{G_\perp^{+1}[\boldsymbol{u}]}\right\}}}$$

But, $K\mathcal{F}^{+1} = \mathcal{F}^{+T}KR$, therefore:

$$= \mathcal{F}^{+T}\overline{\overline{\left\{F_\parallel^{-T}[-\boldsymbol{u}]\,\overline{G_\perp^{+1}[-\boldsymbol{u}]} + F_\perp^{-T}[-\boldsymbol{u}]\,\overline{G_\perp^{+1}[-\boldsymbol{u}]}\right\}}}$$
$$= \mathcal{F}^{+T}\left\{F_\parallel^{-T}[-\boldsymbol{u}]\,\overline{G_\perp^{+1}[-\boldsymbol{u}]} + F_\perp^{-T}[-\boldsymbol{u}]\,\overline{G_\perp^{+1}[-\boldsymbol{u}]}\right\}$$

Image reversal is equivalent to any double transformation in the same direction, hence:

$$= \mathcal{F}^{+T}\left\{\mathcal{F}^{+T}\mathcal{F}^{+T}\left\{F_\parallel^{-T}[\boldsymbol{u}]\right\}\mathcal{F}^{+1}\mathcal{F}^{+1}\left\{\overline{G_\perp^{+1}[\boldsymbol{u}]}\right\}\right.$$
$$\left. +\ \mathcal{F}^{+T}\mathcal{F}^{+T}\left\{F_\perp^{-T}[\boldsymbol{u}]\right\}\mathcal{F}^{+1}\mathcal{F}^{+1}\left\{\overline{G_\perp^{+1}[\boldsymbol{u}]}\right\}\right\}$$

Since the inner transforms of the first term of each product equates to an identity operator and from the operator graph in Figure 2, $\mathcal{F}^{+T}K = \mathcal{F}^{+1}\mathcal{F}^{+1}K\mathcal{F}^{+1}$, therefore:

$$= \mathcal{F}^{+T}\left\{F_\parallel^{+T}[\boldsymbol{u}]\,\overline{G_\perp^{+T}[\boldsymbol{u}]} + F_\perp^{+T}[\boldsymbol{u}]\,\overline{G_\perp^{+T}[\boldsymbol{u}]}\right\}$$

which confirms the the two forms of this correlation equation are indeed equivalent. A similar process can be applied to Eqns (21) and (22).

6 CONCLUSIONS

A careful choice of the direction of the shift component and/or conjugate operation in the definition of the correlation function prior to the derivation of hypercomplex forms is essential, especially if the desired application is sensitive to the sign of the shift as in those using phase information. Although the alternative forms of hypercomplex correlation, derived from the closed operator set, compound the difficulty in deriving and validating the desired form, the use of the operator graph and hypercomplex identities presented simplifies the procedure and adds confidence. The operator algebra presented confirms the existence of the closed set and is equally applicable to the derivation and validation of hypercomplex convolution and/or other hypercomplex processing functions.

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References

[1] T. A. Ell and S. J. Sangwine. Hypercomplex Wiener-Khintchine theorem with application to color image correlation. In *IEEE International Conference on Image Processing (ICIP 2000)*, volume II, pages 792–795, Vancouver, Canada, 11–14 September 2000. Institute of Electrical and Electronics Engineers.

[2] S. J. Sangwine, T. A. Ell, and C. E. Moxey. Vector phase correlation. *Electronics Letters, to appear*, 2001.

[3] R.C. Gonzales and P. Wintz. *Digital Image Processing*. Addison-Wesley, 1987.

[4] H. S. M. Coxeter. Quaternions and reflections. *American Mathematical Monthly*, 53:136–146, March 1946.

[5] R. N. Bracewell. *The Fourier Transform and its Applications*. McGraw-Hill, 2nd edition, 1978.

[6] F. H. Lange. *Correlation Techniques: Foundations and Applications of Correlation Analysis in Modern Communications, Measurements and Control*. Iliffe Books Ltd., London, 1st edition, 1967.