IMPLEMENTATION ISSUES IN 2-D FILTER BANK DESIGN BASED ON MATRIX FACTORIZATION

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ABSTRACT

The problem of the design and effective implementation of multi-dimensional filter banks with prescribed properties is considered. Several algorithms for polyphase matrices factorization are presented.
After such a factorization the number of computations may become much lower. The results for the 2-channel FBs are given. For the 3-channel multirate system an algorithmic version of Suslin's stability theorem may be useful for factoring the polyphase matrices.

1 INTRODUCTION

The growing demand for *processing* and *compression* of still two-dimensional (2-D) images and video (3-D) signals in *telecommunications* and *multimedia technology* motivates the fact that increasingly more attention is being paid to multi-dimensional (M-D) digital filters.

The theory of Gröbner bases for ideals and modules over a multivariate polynomial ring, $\mathcal{K}[z_1, z_2 \dots z_n]$, when $\mathcal{K}$ is an arbitrary but fixed field and $z_1, z_2 \dots z_n$ are independent variables, is applied to solve several problems of interest in multi-dimensional systems and signal processing. An algorithmic proof of Suslin's stability theorem provides a method for finding an explicit factorization of a given polynomial matrix into elementary matrices. Gröbner bases techniques are used in the implementation of the algorithm.

2 DESIGN OF M-D PR LP FILTER BANKS

2.1 Requirements

The usual requirements that M-D FBs should meet are the next:

- perfect reconstruction (PR) property;
- linear phase (LP) property;
- the filters should be FIR;
- nonseparable lattices and FBs are desirable;
- the frequency responses should be quite smooth at the edges of the stop-bands.

Some new results in the theory of FBs design with these properties based on the theory of Gröbner bases were obtained in [1, 2, 3].

An important fact is the next: for FIR filter banks, a synthesis polyphase matrix $\mathbf{F}(\mathbf{z})$ and an analysis polyphase matrix $\mathbf{H}(\mathbf{z})$ which form a perfect reconstruction (PR) pair are guaranteed to exist if and only if $\det(\mathbf{H}(\mathbf{z}))$ is a monomial.

It can be seen that the main difficulty in achieving efficient multirate systems for processing of M-D signals suitable for a wider range of industrial problems lies in the simultaneous fulfillment of all necessary properties.

2.2 Application of Bernstein polynomials

It is assumed that the type of downsampling is quincuncial, which is the simplest nonseparable downsampling lattice [8]. The quincunx sublattice is generated by $V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The PR condition can be written then as $H_0(z_1, z_2)H_1(-z_1, -z_2) - H_1(z_1, z_2)H_0(-z_1, -z_2) = = z_1^{-2k_1-1} z_2^{-2k_2}$, where $H_0(z_1, z_2), H_1(z_1, z_2)$ are the low-pass and high-pass filters of the analysis filter bank and k_1 and k_2 are arbitrary.

As it was shown in [6, 7] the Bernstein polynomials may be applied in order to design the FBs with the properties mentioned above.

In this case the following low-pass analysis filter was found

$$H_0(z_1, z_2) = \frac{1}{2^{4N}} \sum_{i=0}^{N} \sum_{j=0}^{N-i} g_{i,j} \begin{pmatrix} N \\ i \end{pmatrix} \begin{pmatrix} N \\ j \end{pmatrix} (-1)^{i+j}$$

$$(1 - z_1^{-1})^{2i}(1 + z_1^{-1})^{2(N-i)} \cdot (1 - z_2^{-1})^{2j}(1 + z_2^{-1})^{2(N-j)},$$

with $g_{i,j}$ chosen according to the given FB's properties.

The values of N (and M for the high-pass filter) allow one to adjust the smoothness of the frequency responses for the low-pass and high-pass filters.

For the case $N = 1, M = 1$ the polyphase matrices are

$\mathbf{H_p}(a,b) =$

$$= \begin{bmatrix} 1 & 1/4\cdot(1+b)(1+a) \\ 1/4\cdot a(1+b)(1+a) & \begin{matrix} 1/16\cdot(1+2b+2a+b^2- \\ -28ab+a^2+2ab^2+ \\ +2a^2b+a^2b^2) \end{matrix} \end{bmatrix}$$

and

$\mathbf{F_p}(a,b) =$

$$= \begin{bmatrix} \begin{matrix} 1/16\cdot(1+2b+2a+b^2- \\ -28ab+a^2+2ab^2+ \\ +2a^2b+a^2b^2) \end{matrix} & -1/4\cdot(1+b)(1+a) \\ -1/4\cdot a(1+b)(1+a) & 1 \end{bmatrix},$$

where $a = z_1^{-1}, b = z_2^{-1}$.

3 FACTORIZATION OF TWO-CHANNEL 2-D FILTER BANKS

3.1 Factorization of polyphase matrices

Any $M-$channel filter bank is represented by $M \times M$ polyphase polynomial matrices. The polyphase matrix may be factorized into a product of **elementary** and **diagonal** matrices by application of a Gaussian elimination procedure (an elementary matrix $\mathbf{e}_{ij}(f)$ is a matrix which coincides with the identity except for possibly a single off-diagonal entry f in the ij-position). As a result, the following factorization was obtained: $\mathbf{H_p} = \mathbf{H}_1 \cdot \mathbf{H}_2 \cdot \mathbf{H}_3$, where $\mathbf{H_1} = \begin{bmatrix} 1 & 0 \\ 1/4\cdot a(1+b)(1+a) & 1 \end{bmatrix}$, $\mathbf{H_2} = \begin{bmatrix} 1 & 0 \\ 0 & -2\cdot a^2 b \end{bmatrix}$, $\mathbf{H_3} = \begin{bmatrix} 1 & 1/4\cdot(1+b)(1+a) \\ 0 & 1 \end{bmatrix}$.

This example is for the case when $N = 1, M = 1$ (see [6, 7]). Similar results were obtained for $N = M = 2$ and $N = M = 3$. It should be mentioned that this procedure may be applied for any values of N and M.

3.2 Comparison of operations number

The main reasons behind the factorization of the polyphase matrices were:

- to reduce the number of required computations (additions, multiplications),
- to obtain 'good' coefficients (integers, powers of two and so on) for the filters.

The necessary numbers of computations for both - non-factorized and factorized cases - are given in the table 1. It is evident that the factorization of the polyphase polynomial matrices has a really big impact on the computation speed. The fact that the coefficients of the multipliers may be powers of two is also quite important.

In the table * denotes NO factorization and ** - FACTORIZED.

Table 1: **Comparison of non-factorized and factorized polyphase matrices**

N	**M**	*Additions*		*Multiplic.*		*Operations*		**Gain**
		*	**	*	**	*	**	*/**
1	1	14	**6**	8	**3**	22	**9**	**2.44**
2	2	50	**14**	32	**5**	82	**19**	**4.3**
3	3	109	**30**	107	**27**	216	**57**	**3.78**

4 REALIZATION OF THREE- AND MORE CHANNEL MULTIRATE SYSTEMS

4.1 Gröbner bases and an algorithmic version of the Suslin's stability theorem

The factorization technique used above will not work for general 2×2 matrices. However, according to a result from the area of commutative algebra known as Suslin's stability theorem [5], any 3×3 or larger polynomial matrix with determinant one can be factored into a product of elementary matrices. An algorithmic version of Suslin's stability theorem is presented in [4]. Gröbner basis techniques play a role in this algorithm. The algorithm consists of three main steps:

- reduction to the special case of $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where a, b, c and d are multivariate polynomials with $ad - bc = 1$,
- generation of solutions over finitely many suitable local rings (which allows division by certain polynomials),
- patching together the local solutions (which involve ratios of polynomials) to obtain a global solution (which involves strictly polynomials).

Gröbner bases are an important tool in solving problems involving multivariate polynomials. The availability of Buchberger's algorithm for computing Gröbner bases (and computers fast enough to run the algorithm) has been a catalyst for the mathematical theory of multivariate polynomials. Buchberger's algorithm generalized the division algorithm for univariate polynomials and Gaussian elimination for linear polynomials. Gröbner basis applications abound in mathematics, computer science and engineering. The theory of Gröbner bases has become increasingly popular for further development, adaptation and use in electrical engineering. In fact, many interesting applications in various fields of electrical engineering have already been developed.

Unfortunately, the aforementioned algorithm for Suslin's stability theorem is not practical. One reason is the patching step which patches together solutions over local rings in such a way as to obtain a global solution. The Hilbert basis theorem, which states that any ideal of a multivariate polynomial ring over a field is finitely generated, guarantees that only finitely many local solutions are needed to obtain a global solution. However, there is no a priori bound on exactly how many local solutions are necessary.

4.2 Practical example

Can modifications be made to produce practical implementable algorithms? For instance, when the algorithm for Suslin's stability theorem is applied to some matrices, the patching step is not needed; the local case subalgorithm actually yields a global (polynomial) solution. Thus, a solution can be found much more easily than by using the entire algorithm. One question needing further investigation is whether or not the local case subalgorithm will work in enough cases to be of practical use. For an example, consider the matrix

$$M = \begin{bmatrix} 1+g_0+g_1 & (px+q)^4 & 0 \\ (ry+s)^4 & 1-g_0+g_1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where p, q, r, and s are real numbers and $g_0 = \sqrt{2}\,(px+q)\,(ry+s)$, $g_1 = (px+q)^2(ry+s)^2$. The following factorization of M was found by using only the local case subalgorithm:

$$\begin{aligned}
&\mathbf{e}_{21}((ry+s)^4(1-g_0+g_1))\cdot\\
&\mathbf{e}_{23}(-(px+q)^3(ry+s)^4)\cdot\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{32}(1)\cdot\\
&\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{21}\left((px+q)(ry+s)^4(1-g_0+g_1)\right)\cdot\\
&\mathbf{e}_{23}(-(px+q)^3(ry+s)^4)\cdot\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{32}(1)\cdot\\
&\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{21}((px+q)^2(ry+s)^4(1-g_0+g_1)\cdot\\
&\mathbf{e}_{23}(-(px+q)^3(ry+s)^4)\cdot\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{32}(1)\cdot\\
&\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{21}\left((px+q)^3(ry+s)^4(1-g_0+g_1)\right)\cdot\\
&\mathbf{e}_{23}(-(px+q)^3(ry+s)^4)\cdot\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{32}(1)\cdot\\
&\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{21}(1-g_0+g_1)\cdot\mathbf{e}_{12}(-1-g_0-g_1)\cdot\\
&\mathbf{e}_{32}(-1)\cdot\mathbf{e}_{23}(1)\cdot\mathbf{e}_{32}(-1)\cdot\\
&\mathbf{e}_{21}(-(\sqrt{2}(ry+s)+(px+q)(ry+s)^2)\cdot\\
&\mathbf{e}_{12}(px+q)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\\
&\mathbf{e}_{12}(1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{12}(1)\cdot\\
&\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(-\sqrt{2}(ry+s)-(px+q)(ry+s)^2)\cdot\\
&\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{32}(1)\cdot\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{23}(1)\cdot\mathbf{e}_{32}(g_1^2)\cdot\\
&\mathbf{e}_{31}(-(px+q)(1-g_0+g_1))\cdot\mathbf{e}_{32}(-1)\cdot\mathbf{e}_{23}(1)\cdot\\
&\mathbf{e}_{32}(-1)\cdot\mathbf{e}_{21}(-\sqrt{2}(ry+s)+(px+q)(ry+s)^2)\cdot\\
&\mathbf{e}_{12}(px+q)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\\
&\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\\
&\mathbf{e}_{12}(-\sqrt{2}(ry+s)-(px+q)(ry+s)^2)\cdot\mathbf{e}_{23}(-1)\cdot\\
&\mathbf{e}_{32}(1)\cdot\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{23}(px+q)\cdot\mathbf{e}_{32}((px+q)^3(ry+s)^4)\\
&\cdot\mathbf{e}_{31}(-(px+q)(1-g_0+g_1))\cdot\\
&\mathbf{e}_{32}(-1)\cdot\mathbf{e}_{23}(1)\cdot\mathbf{e}_{32}(-1)\cdot\\
&\mathbf{e}_{21}(-\sqrt{2}(ry+s)+(px+q)(ry+s)^2)\cdot\mathbf{e}_{12}(px+q)\cdot\\
&\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{12}(1)\cdot\\
&\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\\
&\mathbf{e}_{12}(-\sqrt{2}(ry+s)-(px+q)(ry+s)^2)\cdot\\
&\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{32}(1)\cdot\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{23}((px+q)^2)\cdot\\
&\mathbf{e}_{32}((px+q)^2(ry+s)^4)\cdot\\
&\mathbf{e}_{31}(-(px+q)(1-g_0+g_1))\cdot\mathbf{e}_{32}(-1)\cdot\mathbf{e}_{23}(1)\cdot\\
&\mathbf{e}_{32}(-1)\cdot\mathbf{e}_{21}(-\sqrt{2}(ry+s)+(px+q)(ry+s)^2)\cdot\\
&\mathbf{e}_{12}(px+q)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\\
&\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1)\cdot\\
&\mathbf{e}_{12}(-\sqrt{2}(ry+s)-(px+q)(ry+s)^2)\cdot\\
&\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{32}(1)\cdot\mathbf{e}_{23}(-1)\cdot\mathbf{e}_{23}((px+q)^3)\cdot\\
&\mathbf{e}_{32}((px+q)(ry+s)^4)\cdot\mathbf{e}_{31}(-(px+q)(1-g_0+g_1))\cdot\\
&\mathbf{e}_{21}(-1)\cdot\mathbf{e}_{12}(1)\cdot\mathbf{e}_{21}(-1).
\end{aligned}$$

5 SUMMARY

The methods for generation of M-D FBs with the desired properties based on polynomial approaches are given.

Bernstein polynomials allow one to design analytically the polyphase polynomial matrices. The factorization of these matrices speeds up the computation rate.

Two types of M-D multirate systems are considered - the 2-channel and 3-channel systems.

Implementation issues are based on construction of factorizations allowing one to obtain effective realizations of multirate systems which are suitable for a wider range of industrial problems.

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