MYRIAD FILTER BASED FORM OF THE DFT

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ABSTRACT

Estimation of the DFT coefficients for signals embedded in an α-stable noise is considered. The DFT form based on the myriad filter is proposed. The optimal value of the myriad filter parameter is determined as a function of the α-stable noise parameters. The proposed form of the DFT is compared with the other robust forms of the DFT in various noisy environments.

1 INTRODUCTION

It is well known that the standard DFT produces poor results in an impulse noise environment. When a signal is embedded in the impulse noise the standard DFT cannot be used for accurate spectral analysis. The standard DFT can be understand as a mean value of the modulated signal's samples. This definition follows from a minimization problem with the loss function in a form of the squared absolute value. The robust M-DFT with the loss function in a form of absolute value function is proposed in [1, 2]. An iterative procedure for the robust M-DFT determination is presented [1, 2]. The median (noniterative) form of the robust DFT is defined by a separate minimization of the real and imaginary parts of the error function in [4]. It behaves similarly as the robust M-DFT calculated by using the iterative procedure. The L-estimation based forms of signal transforms are proposed in [3]. Recently the α-stable noises have been widely used as a model of the impulse noise [5, 6, 7, 8]. Two very important noises: Gaussian and Cauchy noise belong to this class. The myriad filters are proposed for filtering of signals embedded in the α-stable noises [6, 8]. Estimation of the DFT for signals embedded in the α-stable noises based on the myriad filter is the topic of this paper.

The paper is organized as follows. The theoretical background, including properties of the α-stable noises and basic filtering techniques, is presented in Section II. The myriad form of the DFT is developed in Section III. Numerical examples and statistical analysis are given in Section IV. An extension of the presented forms to the time-frequency (TF) analysis is done in Section V.

2 THEORETICAL BACKGROUND

2.1 Alpha-Stable Noise

Signals in communications can often be disturbed by an impulse kind of noise. It can be caused by atmospheric phenomenon and human activities. Modeling of the impulse noise is considered in numerous publications [7, 9]. A widely used model of impulse disturbances is described by the Laplacian probability density function (pdf) $p(\xi) = e^{-|\xi|}/2\pi$. However, the impulse noises that usually appear in practice are of different nature than the Laplacian noise, especially around $\xi = 0$. Namely, they have smooth pdf around $\xi = 0$. This is the reason for introducing the symmetric α-stable noises with smooth pdf around $\xi = 0$. Two very important noises: Gaussian and Cauchy noise, belong to this class. This class of noises is defined by using the characteristic function:

$$\phi(\omega) = FT\{p(\xi)\} = e^{-\gamma|\omega|^{\alpha}} \qquad (1)$$

where $\alpha \in [0, 2]$, while γ is the dispersion, $\gamma > 0$. The parameter α is a measure of the impulse nature of noise. Higher values of α mean lower impulse nature and vice versa. The Gaussian noise follows from (1) for $\alpha = 2$:

$$p(\xi) = \frac{1}{\sqrt{4\pi\gamma}} e^{-\xi^2/4\gamma} \qquad (2)$$

where 2γ is equal to the noise variance. For $\alpha = 1$ the Cauchy noise:

$$p(\xi) = \gamma/[\pi(\gamma^2 + \xi^2)] \qquad (3)$$

is obtained. The pdf for other noises from this class, when $\alpha \neq 1$ and $\alpha \neq 2$, cannot be represented by a closed form expression. The approximative expressions for the pdf for various α are developed in [5].

In order to simulate the α-stable noise we will use the following expression [5]:

$$\nu_\alpha(n) = \begin{cases} \frac{\sin(\alpha \cdot u(n))}{\cos^{1/\alpha}(u(n))} \left[\frac{\cos((1-\alpha)u(n))}{e(n)}\right]^{\frac{1-\alpha}{\alpha}} & \alpha \neq 1 \\ \tan(u(n)) & \alpha = 1 \end{cases}, \qquad (4)$$

where $u(n)$ is a uniformly distributed noise over $[-\pi/2, \pi/2]$, while $e(n)$ is the standard exponential process with the pdf: $p_e(\xi) = e^{-\xi}$, where $\xi \geq 0$. The noises given by (4) are α-stable with $\gamma = 1$, while a noise with $\gamma \neq 1$ follows from (4) as $\nu_{\alpha\gamma}(n) = \gamma^{1/\alpha}\nu_\alpha(n)$.

2.2 Signal Filtering

Consider a signal $f(n)$ corrupted by a white noise $\nu(n)$:

$$x(n) = f(n) + \nu(n). \tag{5}$$

The estimation of $f(n)$ based on the noisy observation $x(n)$ can be obtained by minimizing the functional:

$$J(n;m) = \sum_{k=n-M}^{n+M} F(|x(k) - m|), \tag{6}$$

$$\partial J(n;m)/\partial m|_{m=\hat{f}(n)} = 0, \tag{7}$$

where $\hat{f}(n)$ is an estimate of the signal $f(n)$, while $F(e)$ is the loss function. The loss function of the form $F(e) = -\log p_\nu(n)$, where $p_\nu(e)$ is the pdf function of noise $\nu(n)$, produces the ML estimate of the signal $f(n)$. The ML estimate is "the best" one that can be produced for noisy environment $\nu(n)$. The loss function $F(e) = |e|^2$ in (6) produces the ML estimate for the Gaussian noise (2). It has the form of a moving average filter:

$$\hat{f}(n) = \tfrac{1}{2M+1}\sum_{k=n-M}^{n+M} x(k) =$$

$$= \mathrm{mean}\{x(k)|k \in [n-M, n+M]\}. \tag{8}$$

For the Laplacian noise the ML estimate is obtained for the loss function $F(e) = |e|$, as an output of the median filter:

$$\hat{f}(n) = \mathrm{median}\{x(k)|k \in [n-M, n+M]\}. \tag{9}$$

The myriad filters are solution of the minimization problem (6) for the loss function

$$F(e) = \log(|e|^2 + K^2). \tag{10}$$

The output of the myriad filter is usually obtained through an iterative procedure [6]. The myriad filters are proposed for filtering of signals embedded in the $\alpha-$stable noises. The myriad filter for $K \to \infty$ approaches to the moving average filter, i.e., it is optimal for the Gaussian noise environment ($\alpha = 2$). By decreasing value of K the myriad filter becomes optimal for smaller α, i.e., for noise with a heavy impulse content. The ML estimate for the Cauchy noise ($\alpha = 1$, $\gamma = 1$) produces the myriad filter with $K = 1$. Details on the optimal myriad filter parameters in the α-stable environment can be found in [8].

3 THE MYRIAD FORM OF THE DFT

The standard version of the DFT

$$X(k) = \tfrac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}, \tag{11}$$

can be defined as an average of modulated signal's samples

$$X(k) = \mathrm{mean}\{x(n)e^{-j2\pi nk/N}|n \in [0, N-1]\} \tag{12}$$

for a given k. It can be defined in a similar way as the moving average filter described in Section II.A by using the following minimization problem:

$$J(k;m) = \sum_{n=0}^{N-1} F(x(n)e^{-j2\pi nk/N} - m),$$

$$\partial J(k;m)/\partial m^*|_{m=X(k)} = 0, \tag{13}$$

for $F(e) = |e|^2$. In analogy with the moving average filter the standard DFT exhibits very poor results in an impulse noise environment [1]. This is the reason for introducing the robust M-DFT. It follows from (13) with $F(e) = |e|$. The robust M-DFT in spectral analysis can be understood by considering the analogy with the median filter in signal filtering. Due to the complex nature of the DFT the robust M-DFT cannot be represented in a close form expression. An iterative procedure for its determination is proposed in [1]. The robust M-DFT produces very accurate results for signals corrupted with various impulse noises. However, the robust M-DFT exhibits slightly worse results than the standard DFT in the Gaussian noise environment.

The myriad form of the DFT can be defined by using (13) and the loss function (10). It can improve the DFT accuracy for the α-stable noise environment, in comparison with both the standard DFT and the robust M-DFT. The myriad form of the DFT, $X_K(k)$, where K is the parameter of the loss function (10), can be obtained from (13) in the form:

$$X_K(k) = \frac{\sum_{n=0}^{N-1} \frac{x(n)e^{-j2\pi nk/N}}{|e(n,k)|^2+K^2}}{\sum_{n=0}^{N-1}[|e(n,k)|^2 + K^2]^{-1}}, \tag{14}$$

where $e(n,k)$ is the error function:

$$e(n,k) = x(n)e^{-j2\pi nk/N} - X_K(k). \tag{15}$$

It can be calculated by using the fixed point iterative procedure:

1) The initial guess is the standard DFT (11), $X_K^{(0)}(k) = X(k)$. In heavy impulse environments the robust median based DFT can be used as the initial guess.

2) The next iterations can be calculated as:

$$X_K^{(i)}(k) = \frac{\sum_{n=0}^{N-1} \frac{x(n)e^{-j2\pi nk/N}}{|e^{(i-1)}(n,k)|^2+K^2}}{\sum_{n=0}^{N-1}[|e^{(i-1)}(n,k)|^2 + K^2]^{-1}}, \tag{16}$$

where:

$$e^{(i-1)}(n,k) = x(n)e^{-j2\pi nk/N} - X_K^{(i-1)}(k). \tag{17}$$

3) The stopping rule is:

$$|X_K^{(i)}(k) - X_K^{(i-1)}(k)|/|X_K^{(i-1)}(k)| \leq \eta, \tag{18}$$

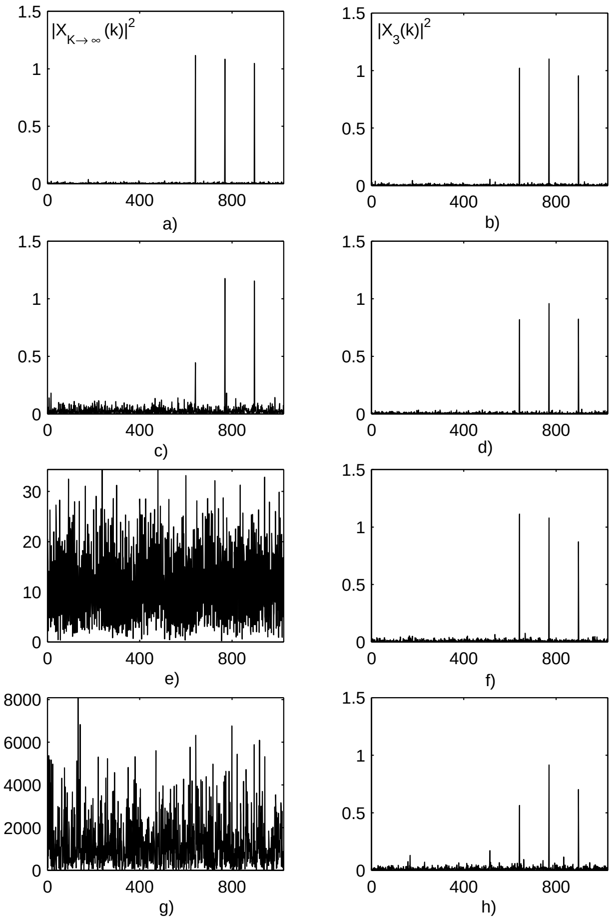

Figure 1: The standard (left column), and the myriad form for $K = 3$ (right column) of the periodogram: a) and b) Gaussian noise ($\alpha = 2$); c) and d) $\alpha = 1.5$; e) and f) Cauchy noise ($\alpha = 1$); g) and h) $\alpha = 0.5$.

where $\eta > 0$ is the accuracy of the procedure (in our statistical analysis and numerical examples $\eta = 0.1$ is used).

The procedure convergence can be proven by using the results from [6]. In analogy with the myriad filter $X_{K\to\infty}(k)$ is equal to the standard DFT.

4 STATISTICAL ANALYSIS

Consider the signal:

$$f(t) = \sum_{l=1}^{3} \exp(jl128\pi t) \tag{19}$$

within the interval $T = 2$, with $N = 1024$ samples. The maximal frequency in the DFT is $s = N\pi/T$. The signal is corrupted by the complex α-stable noise with independent real and imaginary part $\nu'_\alpha(n) + j\nu''_\alpha(n)$, with dispersion $\gamma = 1$. Eight different values of α are considered: $\alpha = r/4$ where $r = 1, 2, ..., 8$. The mean squared error (MSE):

$$MSE = 10\log_{10}\frac{1}{N}\sum_{n=0}^{N-1}|X_K(k) - F_S(k)|^2[dB], \tag{20}$$

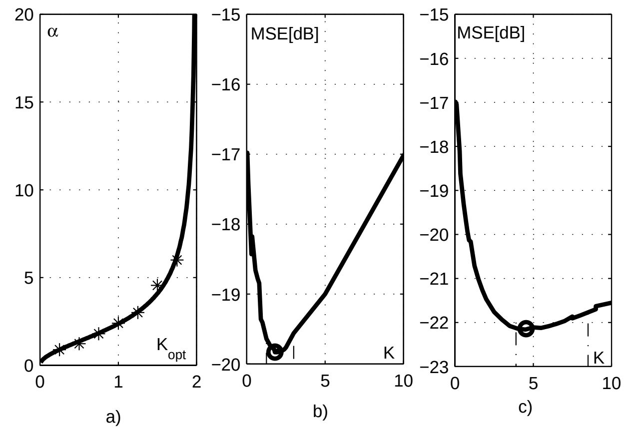

Figure 2: Statistical performance of the myriad form of the DFT: a) Empirical curve $K_{opt} = f(\alpha)$; b) The MSE as a function of K for $\alpha = 0.75$ (circle represents the MSE minimum, dashed lines represent a zone around the minimum with the MSE less than 0.5dB higher than in the optimal point); c) The MSE as a function of K for $\alpha = 1.5$.

is used as a measure of the transform quality, where $F_S(k)$ is the standard DFT of the nonnoisy signal $f(n)$ obtained by sampling $f(t)$. The results of our experiment obtained by using the Monte-Carlo simulation are summarized in Table I. Note that values of the parameter K are considered in the range $K \in [0.00001, 100]$. The following notations has been used: DFT - the standard DFT, M-DFT - the robust M-DFT, $K = 3$ - the myriad form of the DFT with $K = 3$ (in our experiments this form of the DFT is close to the minimal MSE in all examples), Opt. - the myriad form of the DFT that produces the minimal MSE, K_{opt} - values of the K parameter for which the myriad form produces the minimal MSE. The periodograms (squared module of the standard DFT and the myriad form of the DFT for $K = 3$) for several values α are shown in Figure 1. Also, the values of the MSE for signal (19) embedded in a noise with the real and imaginary part given as a cube of the Gaussian noise $\nu_2'^3(n) + j\nu_2''^3(n)$ are shown in Table I - last row (Cub). This noise is used as model of the impulse noise in [1, 2].

From Table I it can be concluded that the value of the optimal parameter K_{opt} decreases as α increases. This can be expected from analysis of the myriad filter performance [8]. The function is shown in Figure 2a. Optimal parameter K_{opt} can be approximated by:

$$K_{opt} \approx 2.36\sqrt{\alpha/(2-\alpha)}. \tag{21}$$

This is a higher value than the one proposed for the myriad filter, $K_{opt} = \sqrt{\alpha/(2-\alpha)}$, in [8]. This difference can be explained by the complex nature of the error function. The MSE as a function of K for $\alpha = 1.5$ and

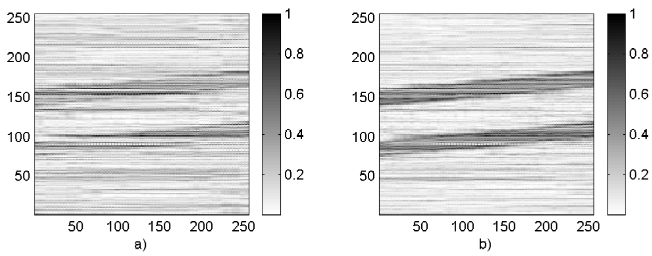

Figure 3: Spectograms of two-component linear FM signal embedded in the Cauchy noise: a) Standard SPEC; b) Myriad form of the SPEC for $K = 3$.

α	DFT	M-DFT	$K = 3$	Opt.	K_{opt}
2	-24.08	-22.64	-23.07	-24.08	100
1.75	-18.54	-22.11	-22.59	-23.06	6
1.5	-12.77	-21.94	-21.93	-22.24	4.55
1.25	3.32	-20.66	-21.38	-21.38	3
1	23.09	-19.77	-20.52	-20.58	2.4
0.75	38.29	-18.05	-19.56	-19.83	1.8
0.5	48.28	-16.67	-18.48	-19.17	1.225
0.25	60.27	-13.37	-17.47	-18.87	0.89
Cub	-15.51	-24.00	-23.88	-24.24	1.4

Table 1: MSE of the DFT estimation: DFT - the standard DFT; M-DFT - the robust M-DFT; $K = 3$ - myriad form of the DFT for $K = 3$; Opt. - myriad form of the DFT producing the minimal MSE; K_{opt} - optimal parameter of the myriad form of the DFT.

$\alpha = 0.75$ is shown in Figures 2b,2c. It can be concluded that for relatively wide region around K_{opt} the MSE remains close to the one produced with K_{opt}.

5 THE MYRIAD FORM OF THE STFT

An extension of the robust M-DFT form to the robust M-STFT is presented in [10]. The myriad form of the STFT can be obtained in a similar way. The implicit form of the myriad STFT can be represented by analogy with (14) as:

$$STFT_K(n,k) = \frac{\sum_{m=0}^{N-1} \frac{x(n+m)e^{-j2\pi km/N}}{|e(n,k,m)|^2+K^2}}{\sum_{m=0}^{N-1}[|e(n,k,m)|^2 + K^2]^{-1}}, \quad (22)$$

where the error function is:

$$e(n,k,m) = x(n+m)e^{-j2\pi km/N} - STFT_K(n,k). \quad (23)$$

A myriad form of the STFT can be calculated by using the similar procedure as in the case of the myriad DFT form (14)-(18). The myriad form of the spectrogram (SPEC) is $SPEC_K(n,k) = |STFT_K(n,k)|^2$.

Example: Consider the signal $f(t) = 2\exp(j24\pi t^2)$ $\cos(64\pi t)$ within an interval $T = 2$, sampled with 512 samples. A rectangular window of length $N = 256$ is used. The signal is corrupted by the complex Cauchy noise with independent real and imaginary part. The standard and myriad SPEC for $K = 3$ are shown in Figure 3. The improvement in TF representation obtained by using the myriad form of the SPEC in comparison with the standard SPEC in the α-stable noisy environment can be easily seen.

6 CONCLUSION

Myriad form of the DFT is presented. It produces better estimates of the nonnoisy DFT in the α-stable noise environment than the standard DFT. Accuracy of this DFT estimate is better than the one by using the robust M-DFT. The accuracy improvement is achieved without increasing the calculation complexity with respect to the robust M-DFT. Statistical analysis is performed by using Monte-Carlo simulations. The extension of the myriad DFT form to the TF analysis is presented, as well.

7 ACKNOWLEDGMENT

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