NOISE-ASSISTED SIGNAL TRANSMISSION FOR NONLINEAR SYSTEMS WITH SATURATION

François CHAPEAU-BLONDEAU, Julio ROJAS VARELA
Laboratoire d'Ingénierie des Systèmes Automatisés (LISA), Université d'Angers,
62 avenue Notre Dame du Lac, 49000 ANGERS, FRANCE.
E-mail : chapeau@univ-angers.fr
web : http://www.istia.univ-angers.fr/~chapeau/

ABSTRACT

We analyze the transmission of a noisy signal by systems which are linear for small inputs and saturate at large inputs. Large information-carrying signals are thus distorted in their transmission. We demonstrate conditions where noise addition to such large input signals can reduce the distortion they undergo in the transmission. This is established for both periodic and aperiodic information-carrying signals. This effect is made possible because the noise acts as a random bias, shifting the operating zone of the nonlinearity, on average, into a region more favorable to the signal transmission. These results constitute a new instance of the nonlinear phenomenon of stochastic resonance where addition of noise may reveal beneficial to the signal.

1 INTRODUCTION

A nonlinear effect, recently introduced under the name of stochastic resonance, displays very attractive potentialities for signal processing. Stochastic resonance establishes that, for certain types of nonlinear coupling between signal and noise, the presence or even the addition of noise, may result in improved performance for the signal (see [1, 2] for recent reviews). Following its introduction some twenty years ago, stochastic resonance has gradually been observed in an increasing variety of nonlinear processes, including electronic circuits [3, 4, 5], optical devices [6, 7], neural systems [8, 9]. It has also progressively been established that stochastic resonance can occur under many different forms, according to the nature of the signal, of the noise, of the nonlinear coupling which are involved and also of the measure of performance receiving improvement from the noise. Various forms of noise-enhanced transmission have been reported for periodic or aperiodic deterministic signals or for random information-carrying signals, in the presence of Gaussian or non-Gaussian, white or colored, noise. Performance has been measured by signal-to-noise ratio, input–output gains, cross-correlation, mutual information, channel capacity, detection probability, estimation efficacy, propagation distance, all these quantities having been shown improvable via noise addition, in specific conditions. So far, systems that have been shown capable of producing a stochastic resonance effect essentially are nonlinear systems with potential barriers or with thresholds. In this case, the essence of the effect is that the information-carrying signal by itself is too small to overcome a threshold or a barrier in the response of the system. Addition of noise then allows some type of cooperation between signal and noise, so as to overcome the threshold or barrier, and elicit a response bearing stronger relation to the signal thanks to assistance from the noise.

In the present paper, we extend the class of nonlinear systems that have been shown capable of stochastic resonance. We consider systems which are purely linear in the small-signal limit (no threshold nor barrier). At the same time, the systems we consider exhibit saturation in their response for large input signals. Large information-carrying input signals are thus distorted in their transmission. We demonstrate conditions where noise addition to such large input signals can reduce the distortion they undergo in the transmission, establishing a new form of noise-improved signal transmission.

2 A NONLINEAR TRANSMISSION

To have a simple demonstration of the new form of stochastic resonance we envision, we consider a deterministic signal $s(t)$ added to a white noise $\eta(t)$ endowed with a probability density function $f_\eta(u)$. The input signal-plus-noise mixture $s(t) + \eta(t)$ is transmitted by a memoryless or static nonlinearity $g(.)$, so as to produce the output signal

$$y(t) = g[s(t) + \eta(t)] \ . \qquad (1)$$

A standard analysis of a stochastic resonance effect would introduce some measure of similarity between input $s(t)$ and output $y(t)$ and investigate the possibility of increasing this measure of similarity through an increase of the noise $\eta(t)$.

When $s(t)$ is periodic with period T_s, output signal $y(t)$ of Eq. (1) generally is a cyclostationary random signal, with a power spectrum containing spectral lines at integer multiples of $1/T_s$ emerging out of a continuous noise background [10]. A standard measure of similarity of $y(t)$ with the T_s-periodic input $s(t)$ is a signal-to-noise ratio defined as the power contained in the output spectral line at $1/T_s$ divided by the power contained in the noise background in a small frequency band ΔB around $1/T_s$.

For the input–output relationship of Eq. (1), the power contained in the output spectral line at frequency n/T_s is given [10] by $|\overline{Y}_n|^2$, where $\overline{Y}_n$ is the order n Fourier coefficient of the T_s-periodic nonstationary output expectation $\mathrm{E}[y(t)]$, i.e.

$$\overline{Y}_n = \left\langle \mathrm{E}[y(t)] \exp\Big(-in\frac{2\pi}{T_s}t\Big) \right\rangle , \tag{2}$$

with the time average defined as

$$\langle ... \rangle = \frac{1}{T_s}\int_0^{T_s} ... \, dt . \tag{3}$$

The output expectation $\mathrm{E}[y(t)]$ at a fixed time t is computable as

$$\mathrm{E}[y(t)] = \int_{-\infty}^{+\infty} g(u) f_\eta[u - s(t)] du . \tag{4}$$

The magnitude of the continuous noise background in the output spectrum is measured [10] by the stationarized output variance $\langle \mathrm{var}[y(t)] \rangle$, with the nonstationary variance $\mathrm{var}[y(t)] = \mathrm{E}[y^2(t)] - \mathrm{E}[y(t)]^2$ at a fixed time t, and

$$\mathrm{E}[y^2(t)] = \int_{-\infty}^{+\infty} g^2(u) f_\eta[u - s(t)] du . \tag{5}$$

A signal-to-noise ratio $\mathcal{R}_n$, for the harmonic n/T_s in the ouput $y(t)$, follows as

$$\mathcal{R}_n = \frac{|\overline{Y}_n|^2}{\langle \mathrm{var}[y(t)] \rangle \, \Delta t \, \Delta B} , \tag{6}$$

where Δt is the time resolution of the measurement (i.e. the signal sampling period in a discrete-time implementation).

When the deterministic input $s(t)$ we seek to extract out of the output $y(t)$ is no longer periodic, then the signal-to-noise ratio $\mathcal{R}_n$ of Eq. (6) is no longer available as a meaningful input–output measure of similarity. Consider $s(t)$ a deterministic aperiodic signal existing over the duration T_s. In such a case, meaningful input–output measures of similarity are provided by cross-correlations as used for instance in [11, 12]. We choose here to use the normalized time-averaged cross-covariance between input $s(t)$ and output $y(t)$; it provides a similarity measure insensitive to both scaling and translation in signal amplitude. We introduce the signals centered around their time-averaged statistical expectation,

$$\widetilde{s}(t) = s(t) - \langle s(t) \rangle \tag{7}$$

and

$$\widetilde{y}(t) = y(t) - \langle \mathrm{E}[y(t)] \rangle , \tag{8}$$

with the time average again defined by Eq. (3). The normalized time-averaged cross-covariance is

$$C_{sy} = \frac{\big\langle \mathrm{E}[\widetilde{s}(t)\widetilde{y}(t)] \big\rangle}{\sqrt{\langle \mathrm{E}[\widetilde{s}^{\,2}(t)] \rangle \, \langle \mathrm{E}[\widetilde{y}^{\,2}(t)] \rangle}} , \tag{9}$$

or equivalently, since $s(t)$ is deterministic,

$$C_{sy} = \frac{\langle s(t)\, \mathrm{E}[y(t)] \rangle - \langle s(t) \rangle \langle \mathrm{E}[y(t)] \rangle}{\sqrt{\big[\langle s(t)^2 \rangle - \langle s(t) \rangle^2\big]\big[\langle \mathrm{E}[y^2(t)] \rangle - \langle \mathrm{E}[y(t)] \rangle^2\big]}} . \tag{10}$$

We now hold two measures of similarity between input $s(t)$ and output $y(t)$, one is $\mathcal{R}_n$ of Eq. (6) for $s(t)$ periodic, the other is C_{sy} of Eq. (10) for $s(t)$ aperiodic. We shall now exhibit conditions for $s(t)$ and the transmission system $g(.)$ where these input–output similarities can be improved when the level of the noise $\eta(t)$ is raised.

3 NOISE-ASSISTED TRANSMISSION

As mentioned above, we investigate here transmission systems $g(.)$ which are linear for small inputs and saturate for large inputs. As a typical example, we consider the nonlinearity

$$g(u) = \tanh(\beta u) \tag{11}$$

with adjustable slope $\beta > 0$, which is linear as βu for small $|u| \ll 1/\beta$ and saturates at ± 1 for large $|u| \gg 1/\beta$.

Further, it is convenient for illustration to consider the case where $\eta(t)$ is a zero-mean uniform noise over $[-\sqrt{3}\sigma_\eta, \sqrt{3}\sigma_\eta]$ with standard deviation σ_η. In this case, with the nonlinearity $g(.)$ of Eq. (11), the integrals of Eqs. (4) and (5) can be evaluated analytically instead of numerically, so as to yield

$$\mathrm{E}[y(t)] = \frac{1}{2\sqrt{3}\beta\sigma_\eta} \ln\left[\frac{\cosh\Big(\beta\big[s(t) + \sqrt{3}\sigma_\eta\big]\Big)}{\cosh\Big(\beta\big[s(t) - \sqrt{3}\sigma_\eta\big]\Big)}\right] \tag{12}$$

and

$$\mathrm{E}[y^2(t)] = \frac{1}{2\sqrt{3}\beta\sigma_\eta}\Bigg[2\sqrt{3}\beta\sigma_\eta + \tanh\Big(\beta\big[s(t) - \sqrt{3}\sigma_\eta\big]\Big) - \tanh\Big(\beta\big[s(t) + \sqrt{3}\sigma_\eta\big]\Big)\Bigg] . \tag{13}$$

Figure 1 shows the output signal-to-noise ratio $\mathcal{R}_1$ at frequency $1/T_s$ from Eq. (6), with $\Delta t \Delta B = 10^{-3}$ as in [13], as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$, for the transmission of the periodic input $s(t) = 10 + 10\sin(2\pi t/T_s)$ by the nonlinearity of Eq. (11). Three values of the slope β are tested.

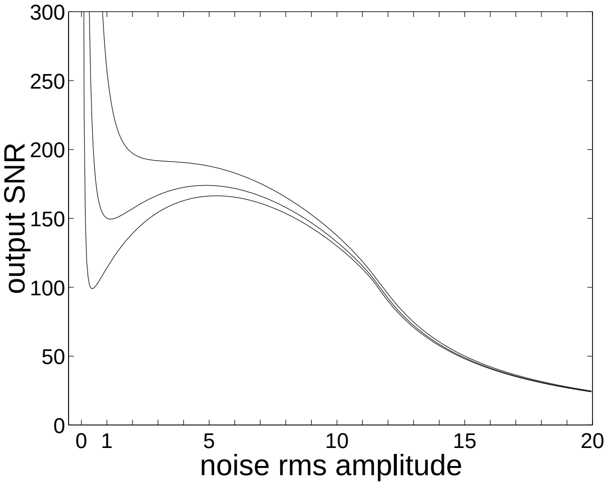

Figure 1: Output signal-to-noise ratio $\mathcal{R}_1$ from Eq. (6) as a function of the rms amplitude σ_η of the uniform noise $\eta(t)$, for $s(t) = 10 + 10\sin(2\pi t/T_s)$ and in Eq. (11) with $\beta = 1$ (top), $\beta = 2$ (middle), $\beta = 5$ (bottom).

In the conditions of Fig. 1, the input $s(t) = 10 + 10\sin(2\pi t/T_s)$ displays excursions to large amplitudes, in relation to the parameter $1/\beta$ of the nonlinearity of Eq. (11). Therefore $s(t)$ is strongly distorted in its transmission. In Fig. 1, at $\sigma_\eta \to 0$, the signal-to-noise ratio $\mathcal{R}_1$ gets infinite because, although the periodic component is very small in the output $y(t)$, the noise component vanishes. Next, as the noise level σ_η increases above zero, $\mathcal{R}_1$ rapidly drops. Yet, when σ_η becomes sufficiently large, $\mathcal{R}_1$ starts to rise. This is properly the stochastic resonance effect. The noise $\eta(t)$ added to the large input $s(t)$ makes it possible to operate the system in regions of the nonlinearity $\tanh[\beta(.)]$ that are more favorable to the transmission of $s(t)$. Thus, on average, the noise reduces the distortion experienced by the large input $s(t)$ in its transmission. This results in a signal-to-noise ratio $\mathcal{R}_1$ in Fig. 1 which can increase as σ_η is raised, to culminate for an optimal noise level where the distortion in the transmission of the periodic component is minimized. This effect of noise-assisted transmission is preserved when β is varied, and, as visible in Fig. 1, is more pronounced for large β when the distortion by the saturating nonlinearity is stronger.

A similar type of stochastic resonance can be obtained in the transmission of an aperiodic signal and measured by an input–output correlation. Figure 2 shows the cross-covariance from Eq. (10), as a function of the rms amplitude σ_η of the zero-mean uniform noise $\eta(t)$, for the transmission by the nonlinearity of Eq. (11) of the aperiodic input

$$s(t) = 5\,\sin\left(2\pi\frac{t}{T_s/2}\right) + 4\,\sin\left(2\pi\frac{t}{3T_s/2}\right) \qquad (14)$$

when $t \in [0, T_s]$, and $s(t) = 0$ elsewhere.

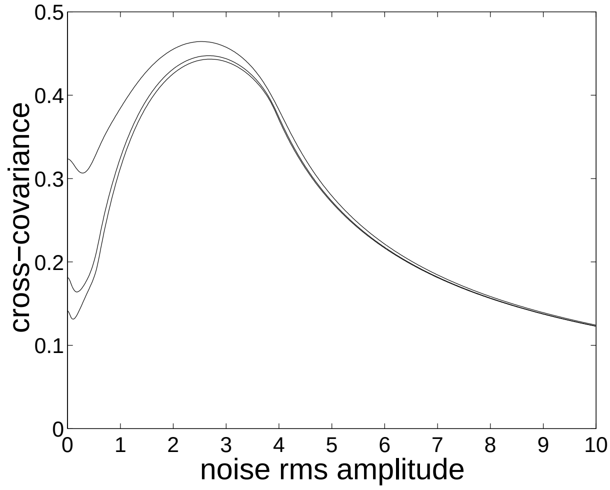

Figure 2: Input–output normalized cross-covariance C_{sy} from Eq. (10) as a function of the rms amplitude σ_η of the uniform noise $\eta(t)$, for $s(t)$ of Eq. (14) and in Eq. (11) with $\beta = 2$ (top), $\beta = 5$ (middle), $\beta = 8$ (bottom).

Again, Fig. 2 illustrates an effect of noise-assisted signal transmission, where the correlation between the aperiodic input $s(t)$ and the output $y(t)$ is maximized for a sufficient nonzero noise level. Figure 3 shows the large signal $s(t)$ of Eq. (14) and the way it is transmitted in the absence of noise and at the optimal noise level. Figure 3(c) displays an ensemble average of the output $y(t)$ showing that noise addition yields an output which is more similar to the input $s(t)$, on average, compared to the transmission with no noise.

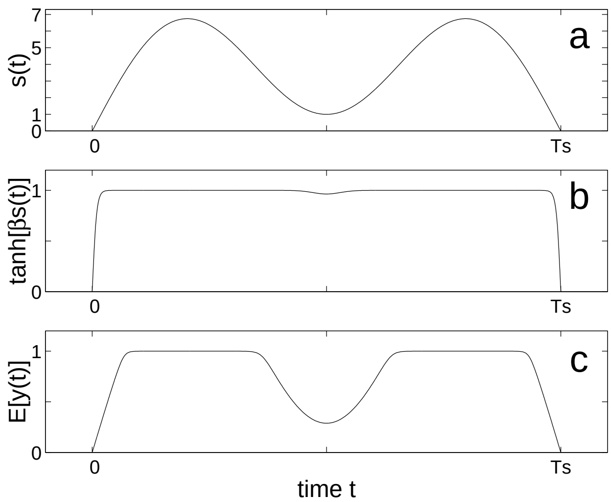

Figure 3: Transmission by Eq. (11) with $\beta = 2$. (a) Input signal $s(t)$ of Eq. (14). (b) Output $y(t) = \tanh[\beta s(t)]$ with no noise. (c) Ensemble average of output $y(t) = \tanh\{\beta[s(t) + \eta(t)]\}$ with $\eta(t)$ a zero-mean uniform noise at the optimum $\sigma_\eta = 2.5$.

An alternative way can be used to quantify the benefit of noise addition. Figure 4 represents the ratio C_{sy}/C_{sx} where $x(t) = s(t) + \eta(t)$ is the input signal-plus-noise mixture, and C_{sx} the normalized cross-covariance between $s(t)$ and $x(t)$ computed as in Eqs. (9)–(10). The ratio C_{sy}/C_{sx} also can be increased by raising the noise, and it culminates at a maximum. The optimal value of the noise is different for C_{sy} of Fig. 2 and for C_{sy}/C_{sx} of Fig. 4, because they are two distinct quantitative measures of a qualitatively similar effect of noise-improved transmission.

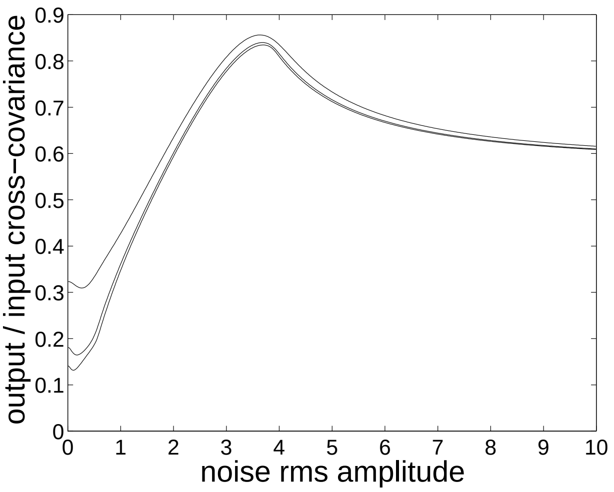

Figure 4: Output/input ratio of the cross-covariance C_{sy}/C_{sx} (see text) as a function of the rms amplitude σ_η of the uniform noise $\eta(t)$, for $s(t)$ of Eq. (14) and in Eq. (11) with $\beta = 2$ (top), $\beta = 5$ (middle), $\beta = 8$ (bottom).

4 CONCLUSION

We have shown that the transmission of a signal by a saturating nonlinearity can be improved by addition of noise. This property has been obtained here with the nonlinearity of Eq. (11) and uniform noise. These conditions are merely illustrative, and it can be verified that the effect is preserved in many other conditions, especially with Gaussian noise. Such an effect can be useful when a signal has to be transmitted by a nonlinear system over which no full control is available, especially to adjust the operating zone of the nonlinearity in accordance with the signal. Here we demonstrated that with large signals, not well positioned in relation to a saturating nonlinearity, addition of noise at the input provides a means of shifting the operating zone of the nonlinear response to a region more favorable to the transmission of the signal. These results can be interpreted as a new instance of the general phenomenon of stochastic resonance, by which, in the presence of nonlinear coupling between signal and noise, addition of noise may reveal beneficial to the signal.

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