ANALYTICAL LINKS BETWEEN STEERING VECTORS AND EIGENVECTORS

Nadège THIRION, Jérôme MARS, Jean-Louis LACOUME

CEPHAG-ENSIEG, BP 46, rue de la Houille Blanche,
38402 ST MARTIN D'HERES Cedex France
Tél/Fax: (33) 76.82.64.21 / (33) 76.82.63.84
e-mail: thirion@cephag.observ-gr.fr

ABSTRACT

We consider the problem of separation of convolutive mixtures of wideband signals impinging on an antenna of sensors focusing on the case of interfering seismic waves. We are looking at the spectral matrix filtering method. The analytical study of its resolving power, makes it possible for us to theoretically justify its use but even to explain its deficiencies in difficult context (waves of very close energies or/and too near slowness for instance). But first, this question induces us to discuss on the links between two basis: the eigenvectors one and the steering vectors one.

1. INTRODUCTION

An increasing interest has been dedicated to the problem of separation of convolutive mixtures of wideband signals impinging on an antenna of sensors. Typical examples can be found in passive sonar, geophysics, and so on... In geophysical operations, the aims of signal processing are the separation and the identification of waves to improve our understanding of the onshore. Many techniques have been developed to achieve these purposes (Karhunen-Loeve transform, f-k and median filter, Spectral Matrix Filtering (*SMF*), τ-p transform, Maximum Likelihood Estimator [4]). We have chosen to focus on the *SMF* method [2,3,6,7]

First, we study the links that exist between two basis: on the one hand the eigenvectors basis which is the mathematical object given by the eigendecomposition of the spectral matrix of the observed signals and on the other hand the steering vectors basis which is the physical object we are interested in. We explain how these two basis fit together. This fitting depends on different parameters, yet, our choice was to express results versus a geometrical criteria (i.e. the spatial coherency of waves vectors) and the energy ratio of the sources.

Then, we are able to analytically determine the resolving power of the *SMF* method, which makes it possible for us to theoretically justify its use but even to explain its deficiencies in difficult context: i.e. waves of very close energies or/and too near slowness.

2. THEORETICAL BACKGROUND

2.1. The model

We assume an antenna composed by N sensors. The signal $r_k(t)$ recorded on the k^{th} sensor is a linear combination of the p detected waves, plus an additive noise [8]. This noise is supposed to be spatially and spectrally white, gaussian and independent of the signals of interest. Its spectral density is noted σ_b^2. These assumptions give in the time domain:

$$r_k(t) = \sum_{\mathbf{i=1}}^{\mathbf{p}} s_{k,i}(t) * a_i(t) + b_k(t) = \sum_{\mathbf{j=1}}^{\mathbf{p}} o_{k,j}(t) + b_k(t) \quad (1)$$

where * is the convolution, $a_i(t)$ is a deterministic amplitude term (referred to as the i^{th} wave-front) which contains no information about the propagation, $s_{k,i}(t)$ describes the propagation of the i^{th} wave recorded on the k^{th} sensor and $b_k(t)$ stands for the noise.

As we opted for a "*multi-narrow band*" approach, the study is performed in the frequency domain: it involves that the calculus made at a given frequency bin does not depend any more on those made at other frequency bins. Using matrix notations, equation (2) is obtained by Fourier Transforming equation (1):

$$R(\nu) = \mathbf{S}(\nu).A(\nu) + B(\nu) = \sum_{j=1}^{p} O_j(\nu) + B(\nu) \quad (2)$$

which can be written:

$$R(\nu) = \underbrace{\mathbf{S}(\nu).\mathbf{D}(\nu)}_{\mathbf{S}'(\nu)}.\underbrace{\mathbf{D}^{-1}(\nu)A(\nu)}_{A'(\nu)} + B(\nu)$$

We use following notations:

- $R(\nu) = FT\big(R(t)\big) = \left[r_1(\nu), \cdots, r_N(\nu)\right]^T$ is the $(N,1)$ vector of the Fourier transforms of the observations. ^{T} is the transposition operation and *FT* the Fourier Transform.
- $O_j(\nu) = S_j(\nu).a_j(\nu)$ is the j^{th} wave vector.
- $\mathbf{S}(\nu) = \left[S_1(\nu), \cdots, S_p(\nu)\right]$ is a (N,p) matrix whose k^{th} column is the so-called k^{th} *steering vector* expressed as: $S_k(\nu) = \left[s_{1,k}(\nu), \cdots, s_{N,k}(\nu)\right]^T$.

The phase of its first component is assumed to be null which implies that the first sensor is chosen as a reference. This convention ensures the unity of the sources. Besides, these steering vectors describe propagation on the antenna. Under the plane waves assumption with neither attenuation nor dispersion, the complex gain between two sensors reduces to a pure phase term. Such an hypothesis is commonly used in Array Processing. Besides, if the antenna is assumed to be linear with regularly spaced sensors, the propagation delay between the 1^{st} and the m^{th} sensor of the k^{th} wave-front (impinging from the direction Θ_k) is simply given by:

$$\tau_{m,k} = m.\Delta.\sin(\Theta_k) / c = m.\tau_k = m\Phi_k \,/\, 2\pi\nu$$

where c is the sound propagation velocity and Δ the distance between two adjacent sensors. Thus, expression of k^{th} steering vector simplifies to:

$$S_k(\phi_k(\nu)) = \left[1, e^{-j\phi_k} \cdots, e^{-j(N-1)\phi_k}\right]$$

There only remain *p* unknown parameters and the propagation matrix has a *Vandermonde* structure. However, in the general case, it is necessary to take into account more complex phenomena to have a realistic model.

We are looking at the problem of separation of colored and coherent sources *A* (because of multi-paths propagation problems). This fact justifies the introduction of a diagonal normalization matrix $\mathbf{D}$ which ensures spectral whitening of the sources i.e. the spectral matrix of the new sources $A'(\nu)$ is given by:

$$E\left[A'(\nu).A'^H(\nu)\right]=\begin{pmatrix}1 & \alpha_{j,i}^* & \cdots \\ \alpha_{i,j}^* & 1 & \\ \vdots & \ddots & \ddots\end{pmatrix} \quad \alpha_{j,i}^*=\alpha_{i,j}=\frac{E\left[a_i.a_j^*\right]}{E\left[a_i.a_i^*\right].E\left[a_j.a_j^*\right]}$$

where $E[.]$ is the mathematical expectation.

In the same way, correlation of sources also has to be taken into account in the equation statement by means of a matrix **T**. This matrix ensures spatial whitening of the sources and satisfies the two following conditions:

(i) $R(\nu)=\underbrace{\mathbf{S}'(\nu).\mathbf{T}^{-1}(\nu)}_{\mathbf{S}''(\nu)}.\underbrace{\mathbf{T}(\nu)A'(\nu)}_{A''(\nu)}+B(\nu)$

(ii) $E\left[A''(\nu).A''^H(\nu)\right]=\mathbf{I}_p \Leftrightarrow E\left[A'(\nu).A'^H(\nu)\right]=\mathbf{T}^{-1}(\nu).\mathbf{T}^{-H}(\nu)$

(3)

where $\mathbf{I}_p$ is the (p,p) identity matrix.

Whatever $\mathbf{S}''(\nu)$, matrix of the new steering vectors, its Singular Values Decomposition (*SVD*) is given by [11]:

$$\mathbf{S}''(\nu)=\mathbf{V}(\nu).\Delta^{1/2}(\nu).\mathbf{\Pi}(\nu) \quad (4)$$

where **V** is a unitary (N,N) matrix (i.e. $\mathbf{V}.\mathbf{V}^H=\mathbf{I}_N$; ^{H} denotes transconjugaison operator), Δ is a (N,p) diagonal matrix whose N-p last lines are null (it is obviously supposed that $N>p$), $\mathbf{\Pi}$ is a (p,p) unitary matrix, which is parametered in our case as a product of Givens rotation matrices ($\mathbf{\Pi}'$), multiplied by a diagonal matrix of pure phase terms (**P**) [11]. In the most simple case which is the two waves case, its expression simplifies to:

$$\mathbf{\Pi}(\theta,\kappa,\psi_1,\psi_2)=\mathbf{\Pi}'.\mathbf{P}=$$
$$\begin{pmatrix}\cos\theta(\nu) & \sin\theta(\nu).e^{j\kappa(\nu)} \\ -\sin\theta(\nu).e^{-j\kappa(\nu)} & \cos\theta(\nu)\end{pmatrix}.\begin{pmatrix}e^{j\psi_1(\nu)} & 0 \\ 0 & e^{j\psi_2(\nu)}\end{pmatrix} \quad (5)$$

It depends on four parameters which obviously vary with the frequency.

Finally, introducing the expressions found in equations (3) and (4) in equation (2), we find:

$$R(\nu)=\mathbf{V}(\nu).\Delta^{1/2}(\nu).\mathbf{\Pi}(\nu).\mathbf{T}(\nu).\mathbf{D}^{-1}(\nu)A(\nu)+B(\nu) \quad (6)$$

This expression, which is the most general one, involves a large number of matrices. Yet, we have proved in [10] that the equation statement that effectively has to be considered depends on another stage which is the estimation of the spectral matrix $\mathbf{\Gamma}(\nu)$ related to the seismic traces. In a practical case of treatment of seismic data, the mathematical expectation is replaced by specific operators, noted $\xi(.)$, like spatial or frequency smoothing [9]. Their purpose is to diminish the influence of the terms that are due to the interactions between different sources, making the inversion of the spectral matrix possible. If smoothing operators do not allow a sufficient decorrelation of the waves, the equation statement is the one given by equation (6) which means even in the most simple case (two waves case) a great complexity (6 parameters still have to be estimated after second order whitening [10] to recover true waves vectors). On the contrary, a satisfying decorrelation of waves leads to the same result as the one obtained considering matrix **T** as a unitary one: the complexity of the problem considerably decreases, and it is even possible to recover the parametrisation classically used in blind separation of independent sources [5]:

$$R(\nu)=\mathbf{V}(\nu).\Delta^{1/2}(\nu).\mathbf{\Pi}(\nu).\mathbf{D}^{-1}(\nu)A(\nu)+B(\nu) \quad (7)$$

2.2. Determination of V and Δ

The spectral matrix of the observations is defined by:

$$\mathbf{\Gamma}(\nu)=\xi\left[R.R^H\right]=\begin{cases}\xi\left[\mathbf{S}''.A''.A''^H.\mathbf{S}''^H\right]+\xi\left[B.B^H\right] \\ \mathbf{V}.\left(\Delta+\sigma_b^2.\mathbf{I}_N\right).\mathbf{V}^H\end{cases} \quad (8)$$

Equation (8) is obtained by reintroducing the parametrisation of *R* that was given in equation (6). It can also be identified with the eigendecomposition of the spectral matrix because of the uniqueness of this one. Thus, eigendecomposition enables the determination of two of the matrices that are looked for: the *p* first columns of matrix **V** are the *p* first eigenvectors of matrix $\mathbf{\Gamma}$ (assuming that eigenvalues have been arranged in a descending way). In the same way, the *p* largest eigenvalues λ_k of $\mathbf{\Gamma}$ are related to Δ. In fact, we have:

$$\Delta=\begin{pmatrix}\sqrt{\lambda_1-\hat{\sigma}_b^2} & 0 & \cdots \\ 0 & \sqrt{\lambda_2-\hat{\sigma}_b^2} & \\ \vdots & \ddots & \ddots\end{pmatrix} \quad (9)$$

The eigenvectors associated with the *p* largest eigenvalues belong to the same subspace (called the Signal Subspace (*SS*)) as the one spanned by the *p* steering vectors of the desired waves. Yet, nothing guarantees the exact fitting between these two basis. This is obviously due to the fact that other matrices involved in the equation statement (among which the unitary matrix $\mathbf{\Pi}$) are not reachable by this own treatment. We can even notice that eigenvectors define an orthonormal basis whereas steering vectors are not necessarily orthogonal.

In next section, we explain how the two basis fit together, and we quantify resolving power of the spectral matrix filtering. The analytical calculations prove that, in most cases, treatments based on exploitation of second order properties of received signals are not sufficient to separate waves but enable extraction of the most energetic one. To reach separation, treatments have to be completed. In the case of blind separation of wideband independent sources, it means that matrix $\mathbf{\Pi}$ has to be estimated: this is achieved by using the fact that this matrix leads to most independent sources [1,5] in the sense of a higher order criteria. In the case of sufficient decorrelation of waves (the equation statement is given by eq. (7) instead of eq. (6)), blind separation of seismic waves has been performed replacing this criteria by a local distance stationarity criteria applied on the phases of the estimated wave vectors [10].

3. ANALYTICAL STUDY OF SPECTRAL MATRIX FILTERING

We focus on the case of two plane waves. The two vectors $V_1(\nu)$ and $V_2(\nu)$ associated with the two largest eigenvalues λ_1 and λ_2 have to be analytically calculated. To reach this purpose, we exploit the two following properties: these vectors are eigenvectors of matrix $\mathbf{\Gamma}(\nu)$ (equation 10) and they are linear combination of steering vectors because of their belonging to the *SS* (equation 11):

$$\Gamma(\nu).V_k(\nu)=\lambda_k.V_k(\nu) \qquad \forall k=1...2$$
$$=\left(S_1'.S_1'^H+S_2'.S_2'^H+\alpha_{1,2}^*S_2'.S_1'^H+\alpha_{1,2}S_1'.S_2'^H+\sigma_b^2.\mathbf{I}_N\right).V_k(\nu) \qquad (10)$$

$$\begin{cases} V_1(\nu)=c_1S_1'+c_2S_2' \\ V_2(\nu)=d_1S_1'+d_2S_2' \end{cases} \qquad (11)$$

where c_1, c_2, d_1, d_2 are complex numbers.
This set of hypothesis leads to the following system:

$$\begin{cases} c_1(\lambda_1-\sigma_b^2-Pa_1-\alpha_{1,2}.S_2'^H.S_1')=c_2\left(S_1'^H.S_2'+\alpha_{1,2}Pa_2\right) \\ c_2(\lambda_1-\sigma_b^2-Pa_2-\alpha_{1,2}^*.S_2'^H.S_1')=c_1\left(S_2'^H.S_1'+\alpha_{1,2}^*.Pa_1\right) \end{cases} \qquad (12)$$

where $Pa_i = S_i'^H.S_i' = \| S_i' \|^2$. The whole calculus is presented in the most simple case: we suppose that the decorrelation stage has been reached ($\alpha_{i,j}=0$).

To solve the system given by equation (12), different cases have to be distinguished:

(i) Waves are geometrically orthogonal (i.e. $S_1'.S_2'^H=0$) but sources have different energies, then the eigenvector which is associated with the largest eigenvalue is collinear to the steering vector of the most energetic wave, and the eigenvector associated with the second eigenvalue is collinear to the steering vector of the less energetic wave. This appears in equation (13). The treatment is completed at the end of the second order stage to the extent that the found basis already coincides with the wanted basis :

$$\begin{cases} \lambda_1=\sigma_b^2+Pa_1 \;;\; V_1=(1/\sqrt{Pa_1}).S_1' \\ \lambda_2=\sigma_b^2+Pa_2 \;;\; V_2=(1/\sqrt{Pa_2}).S_2' \end{cases} \qquad (13)$$

(ii) The case of orthogonal waves with the same energy is a singular one. Eigenvalues are found to be always identical. Whatever the vector belonging to the space spanned by steering vectors, it is an eigenvector. The system always remains undetermined...

(iii) We now suppose that the waves are not orthogonal. It can be easily established that the two largest eigenvalues of the spectral matrix are given by :

$$\lambda_{1/2}=\frac{1}{2}.\left(Pa_1+Pa_2\pm\sqrt{\left(Pa_1-Pa_2\right)^2+4.\left|S_2'^H.S_1'\right|^2}\right)+\sigma_b^2$$

A condition about c_1, c_2 is deduced :

$$c_1/c_2=\frac{2.S_1'^H.S_2'}{\left(Pa_2-Pa_1\right)+\sqrt{\left(Pa_1-Pa_2\right)^2+4.\left|S_2'^H.S_1'\right|^2}}$$

We obtain the same kind of relation for d_1, d_2. These two ratios are representative of the geometrical organization between the two considered basis. The transformation which ensures the passing from one basis to the other one is the multiplication by a compression matrix (Δ) and a unitary matrix expressed as a complex rotation matrix (Π). In the two waves case, it becomes:

$$\mathbf{S}'(\nu)=\mathbf{V}.\Delta^{1/2}.\mathbf{\Pi}=$$
$$\begin{pmatrix}V_1 & V_2\end{pmatrix}.\begin{pmatrix}\sqrt{\lambda_1-\sigma_b^2} & 0 \\ 0 & \sqrt{\lambda_2-\sigma_b^2}\end{pmatrix}\begin{pmatrix}\cos\theta.e^{j.\psi_1} & \sin\theta.e^{j.(\psi_2+\kappa)} \\ -\sin\theta.e^{j.(\psi_1-\kappa)} & \cos\theta.e^{j.\psi_2}\end{pmatrix}$$

Conditions on coefficients c_1, c_2, d_1, d_2 are deduced from this last equality:

$$c_2/d_2=\sqrt{(\lambda_2-\sigma_b^2)/(\lambda_1-\sigma_b^2)}.\tan\theta.e^{-j\kappa}$$
$$c_1/d_1=\sqrt{(\lambda_2-\sigma_b^2)/(\lambda_1-\sigma_b^2)}.\frac{1}{\tan\theta}.e^{-j(\kappa+\pi)} \qquad (14)$$

Thus we have to parameter the unknowns. Uniqueness of this parametrisation is ensured by the normalisation of the eigenvectors:

$$\arg(c_1)+\psi_1=0$$
$$\arg(d_2)+\psi_2=0 \qquad \kappa=\arg(d_2)-\arg(c_2)$$

and :

$$\left.\begin{aligned}|c_1|=\cos\theta/\sqrt{\lambda_1-\sigma_b^2} \\ |c_2|=\sin\theta/\sqrt{\lambda_1-\sigma_b^2}\end{aligned}\right\}\Rightarrow\left|c_2/c_1\right|=\tan\theta$$
$$\left.\begin{aligned}|d_1|=\sin\theta/\sqrt{\lambda_2-\sigma_b^2} \\ |d_2|=\cos\theta/\sqrt{\lambda_2-\sigma_b^2}\end{aligned}\right\}\Rightarrow\left|d_2/d_1\right|=1/\tan\theta$$

We now quantify the dependency of angles θ and κ of the unitary matrix on parameters of interest. In our case, the two desired angles are expressed versus E the energy ratio of the sources ($E=Pa_2/Pa_1$) and ρ the spatial coherency between the two waves. ρ is the normalized scalar product between steering vectors (it is a geometrical criteria). In the case of plane waves, with equispaced sensors (see model given in §2.1), we have:

$$\rho(\nu)=\frac{S_1'^H(\nu).S_2'(\nu)}{\sqrt{\|S_1'(\nu)\|^2.\|S_2'(\nu)\|^2}}=\frac{1}{N}.\frac{\sin(N.\Delta\Phi(\nu))}{\sin(\Delta\Phi(\nu))}.e^{j(N-1)\Delta\Phi(\nu)}$$

with: $\Delta\Phi=\frac{\Phi_1-\Phi_2}{2}$

The module of the spatial coherency varies between 0 and 1 (see figure 1); ρ = 0 for geometrically orthogonal waves. It becomes true if the number of sensors is great and the angles of arrival are different; ρ = 1 for collinear waves.

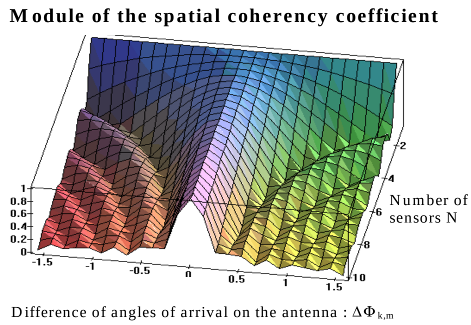

Figure 1: module of the spatial coherency coefficient

Finally, we find that:

$$\left|c_2/c_1\right|=\tan\theta=\frac{(E-1)+\sqrt{(1-E)^2+4.|\rho|^2.E}}{2.|\rho|.\sqrt{E}} \quad \text{and}$$
$$\kappa=\psi_1-\psi_2-(N-1).\Delta\Phi_{2,1}$$

It is also possible to get the expression of eigenvectors, which will make it possible to quantify the resolving power of the spectral matrix. We have established that:

$$\frac{\text{Power of } S_1 \text{ on } V_1}{\text{Power of } S_2 \text{ on } V_1} = \frac{1}{(\tan\theta)^2} = \frac{\text{Power of } S_2 \text{ on } V_2}{\text{Power of } S_1 \text{ on } V_2}$$

Waves of identical energy characterize a singular case because angle θ does not depend on spatial coherency any more. It remains equal to 45° (figure 2). Moreover it is the less favorable one in terms of separation to the extent that, after the second order stage, sources still remain totally mixed (the same proportion of each source on both whitened signals (figure 3)). In the case of orthogonal waves (spatial coherency coefficient equals 0), angle θ remains equal to 0° (separation is achieved after simple projection onto eigenvectors). In all other cases, the separation is still not performed after the second order stage, but on the first eigenvector : proportion of the most energetic source is widely superior to the proportion of the least energetic source. In spite of the fact that second source is less energetic, its proportion remains superior to the proportion of most energetic source, as far as the second eigenvetor is concerned.

4. CONCLUSION

In this work we explain how the basis of steering vectors and eigenvectors fit together and how this fitting depends on different parameters such as the energy ratio of waves and their spatial correlation degree. This study makes it possible for us to justify the use of the *SMF* method in the case of seismic waves with different energies, and to explain the deficiencies of this method in the case of waves of close energies.

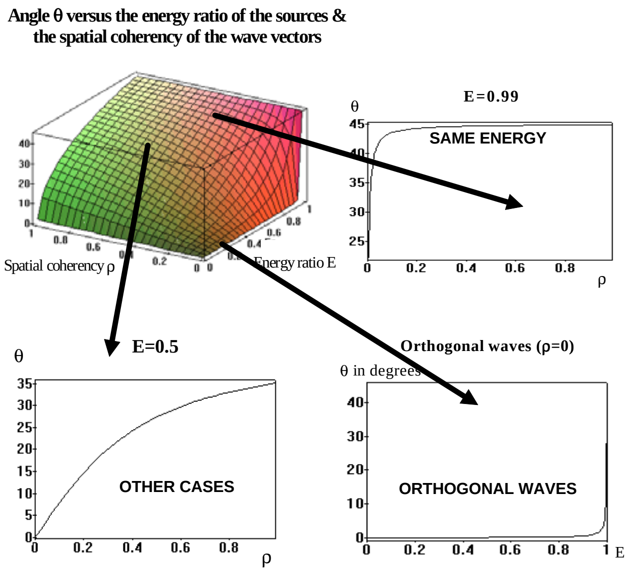

Figure 2: Variations of the angle θ versus energy ratio and spatial coherency of the waves

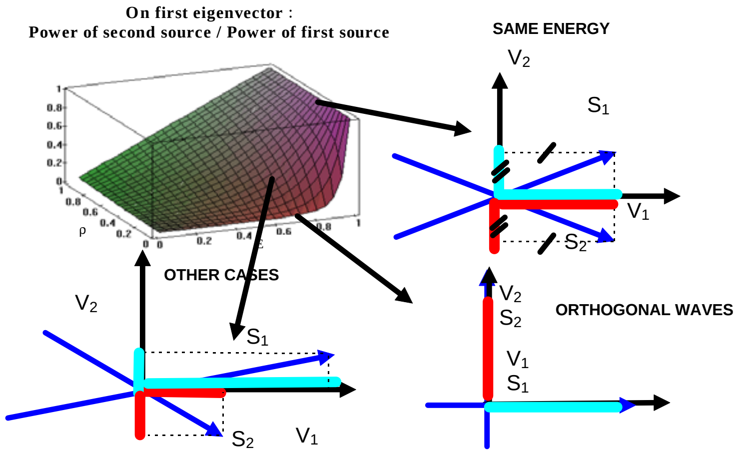

Figure 3: Fluctuations versus spatial coherency and energy ratio

5. REFERENCES

[1] **Comon, P.,** 1989, "Separation of sources using higher order cumulants": *SPIE Conf. On Advanced Algorithms and Architectures for Signal Processing, Real-Time Signal Processing*, vol. XII, pp. 170-181, San Diego, CA.

[2] **Freire, S., Ulrych, L.,** 1988, "Application of singular value decomposition to vertical seismic profiling": *Geophysics*, n° 53, pp. 778-785.

[3] **Glangeaud, F., Mari, J. L., Lacoume J. L.,** juin 1989, "Estimation de la matrice spectrale de signaux certains: Application à la séparation d'ondes en sismique": *Douzième colloque du GRETSI*.

[4] **Haykin, S., 1991**, "Advances in spectrum analysis and array processing": *Prentice-Hall, advanced reference series engineering*, vol. 1& 2.

[5] **Lacoume, J. L., Ruiz, P.,** 1988, "Sources identification : a solution based on the cumulants" : *IEEE ASSP, Workshop on Spectrum Estimation*, Minneapolis, pp. 199-203.

[6] **Mars, J., Glangeaud, F., Lacoume, J. L., Fourmann, J. M., Spitz, S.** 1987, "Séparation of seismic waves": *55th Annual SEG Meeting*, New Orleans, pp. 489-492.

[7] **Mermoz, H.,** 1969, "Elimination des brouilleurs par traitement optimal d'antenne": *Annales des Télécommunications*, tome 24, n°7-8, pp. 282-293.

[8] **Robinson, E. A.,** 1967, "Predictive decomposition of time series with application to seismic exploration": *Geophysics*, vol. 32, pp. 418-484.

[9] **Shan, T. J., Wax, M., Kailath, T**., 1985, "On spatial smoothing for direction of arrival estimation of coherent signals": *IEEE Transactions on Acoustic Speech and Signal Processing*, vol. 33, pp. 806-811.

[10] **Thirion, N**., 1995, "Séparation d'ondes en prospection sismique": *thèse de doctorat l'INPG*.

[11] **Wilkinson, J. H.,** 1965, "The algebraic eigenvalue problem": *Oxford University Press*.