FUZZY CENTER WEIGHTED MEDIAN FILTERS

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ABSTRACT

Stack filters are a class of nonlinear filters, first introduced by Wedent et. al. Stack filters perform well in many situations where linear filters fail. Stack filters include rank order filters, morphological filters and weighted median filters. The stack filter is defined by a Boolean function. The output of Boolean functions is restricted two values (i.e., "0" or "1"). Intuitively, one would expect better performance for stack filters, if the output of Boolean functions is defined from 0 to 1 continuously. We call this Boolean functions fuzzy Boolean functions. We discuss about fuzzy center weighted median (FCWM) filters which is one of the simplest fuzzy stack filters in this paper. Two design methods are shown in this paper.

1 INTRODUCTION

Stack filters [4] were proposed as a generalization of rank order filters in an effort to increase the variety of available nonlinear operations. A Stack filter is a sliding window nonlinear filter whose output at each window position is the result of a superposition of the outputs of a stack of positive Boolean functions operating on thresholded versions of the samples in the filter's window. The output of Boolean functions is restricted two values (i.e., "0" or "1"). Intuitively, one would expect better performance for stack filters, if the output of Boolean functions is defined from 0 to 1 continuously. We call this Boolean functions fuzzy Boolean functions which will be introduced in this paper.

The weighted median (WM) filter, which weights each signal sample within the filter window by some non-negative weight, is an extension of the median filter [1]. It was shown that WM smoothers are quite robust for different types of noise and, at the same time, are able to preserve edges. Center weighted median (CWM) filters[2],[3], which give more weight only to the center value of each window, are subclass of WM filter. It was shown that the behavior of CWM filters can be controlled by center weight to provide good detail preservation or good noise attenuation. WM filters are nonlinear filters belonging to the class of stack filters[4].

In order to expand the class of the CWM filter, fuzzy CWM (FCWM) filters are proposed in this paper. The FCWM filter operation is defined only in the binary domain. The FCWM filter is one of the simplest fuzzy stack filters which are constructed by fuzzy Boolean function.

Experimental results in image restoration demonstrate that FCWM filters give better results than CWM filters, especially in the cases of image is corrupted by non-impulsive noise.

2 THE FCWM FILTER AND ITS DESIGN METHOD

2.1 Threshold Decomposition and CWM Filters in the Binary Domain

Consider a noisy filter input vector $\mathbf{x}(i)$ which is given by

$$\mathbf{x}(i) = [x(i-N),\ldots,x(i),\ldots x(i+N)] \quad (1)$$

Threshold decomposition an M-valued signal $x(i)$ $(0 \le x(i) \le M-1)$ means decompose it into M-1 binary signals $x^{(1)}(i),\cdots,x^{(M-1)}(i)$ according to the thresholding rule:

$$x^{(m)}(i) = \begin{cases} 1 : \mathit{if}\ x(i) \ge m \\ 0 : \mathit{otherwise} \end{cases} \quad (2)$$

$\mathbf{x}^{(m)}(i)$ is also given by

$$\mathbf{x}^{(m)}(i) = [x^{(m)}(i-N),\ldots,x^{(m)}(i),\ldots,x^{(m)}(i+N)]. \quad (3)$$

Median and CWM filters operation can be realized by stack filters operation. Let $R^{(m)}(i)$ be defined as

$$R^{(m)}(i) = (\textit{The number of "1" in the vector } \mathbf{x}^{(m)}(i)\,)\ /\ (2N+1) \quad (4)$$

Each median and CWM operation on binary samples reduces to simple Boolean operation. The median filter is shown as

$$f_{MED}(\mathbf{x}^{(m)}(i)) = f_{MED}(R^{(m)}(i)) = \begin{cases} 1: & if \;\; R^{(m)}(i) > 0.5 \\ 0: & otherwise \end{cases} \quad (5)$$

The output of the median filter is given by

$$y_{MED}(i) = \sum_{m=1}^{M-1} f_{MED}(R^{(m)}(i)) \quad (6)$$

If we consider the Boolean functions in the case of $x^{(m)}(i)=0$ and $x^{(m)}(i)=1$, separately, we can easily extend the Boolean function of the median filter to that of CWM filter as follows:

- In the case of $x^{(m)}(i)=0$

$$f_{CWM}^{(0)}(R^{(m)}(i)) = \begin{cases} 1: R^{(m)}(i) > (N+W_0/2)/(2N+1) \\ 0: R^{(m)}(i) \le (N+W_0/2)/(2N+1) \end{cases}$$

- In the case of $x^{(m)}(i)=1$

$$f_{CWM}^{(1)}(R^{(m)}(i)) = \begin{cases} 1: R^{(m)}(i) > (N+1+W_0/2)/(2N+1) \\ 0: R^{(m)}(i) \le (N+1+W_0/2)/(2N+1) \end{cases} \quad (7)$$

where W_0 is the weight of center sample $x(i)$. $f_{CWM}^{(0)}(R^{(m)}(i))$ and $f_{CWM}^{(1)}(R^{(m)}(i))$ are the Boolean functions in the case of $x^{(m)}(i)=0$ and $x^{(m)}(i)=1$, respectively. The output of the CWM filter is given by

$$y_{CWM}(i) = \sum_{m=1}^{x(i)} f_{CWM}^{(1)}(R^{(m)}(i)) + \sum_{m=x(i)+1}^{M-1} f_{CWM}^{(0)}(R^{(m)}(i)) \quad (8)$$

2.2 FCWM Filters

The output of the Boolean function is restricted to binary values (i.e., 0 or 1). However, one would expect better performance for stack filters, if the output of the Boolean function has the real value between 0 and 1. We call this Boolean functions fuzzy Boolean functions. We propose FCWM filters which are defined by the fuzzy Boolean function and give two types of design methods of those. The fuzzy filter is included in FCWM filters.

2.2.1 Fuzzy Boolean function is derived by way of fuzzy rules: FCWM-1

We derive the fuzzy Boolean functions of FCWM filters $f_{FCWM}^{(0)}(R^{(m)}(i))$ (in the case of $x^{(m)}(i)=0$) and $f_{FCWM}^{(1)}(R^{(m)}(i))$ (in the case of $x^{(m)}(i)=1$) by way of fuzzy rules. Fuzzy rules (p=1,...,P) can be shown as

- In the case of $x^{(m)}(i)=0$
  if $R^{(m)}(i)$ is A_p then $f_{FCWM}^{(0)}(R^{(m)}(i))$ is $\omega_p^{(0)}$
- In the case of $x^{(m)}(i)=1$
  if $R^{(m)}(i)$ is A_p then $f_{FCWM}^{(1)}(R^{(m)}(i))$ is $\omega_p^{(1)}$

where A_p is common fuzzy set of $R^{(m)}(i)$ in the case of $x^{(m)}(i)=1$ and $x^{(m)}(i)=0$. $\omega_p^{(0)}$ and $\omega_p^{(1)}$ are output singletons in the case of $x^{(m)}(i)=1$ and $x^{(m)}(i)=0$, respectively. Fuzzy Boolean functions of the FCWM-1 filter are given by

$$f_{FCWM}^{(*)}(R^{(m)}(i)) = \frac{\sum_{p=1}^{P} m_{A_p}(R^{(m)}(i)) \cdot \omega_p^{(*)}}{\sum_{p=1}^{P} m_{A_p}(R^{(m)}(i))} \quad (9)$$

where $m_{A_p}(R^{(m)}(i))$ is the membership function of fuzzy sets A_p. The flexibility of fuzzy Boolean functions is depend on the number and shape of fuzzy sets. Anyway, the large class of nonlinear functions is able to be generated by this method.

The CWM filter can be expressed by the following if-then rules:

- In the case of $x^{(m)}(i)=0$

if $R^{(m)}(i)$ is *Small* then $f_{FCWM}^{(0)}(R^{(m)}(i))$ is $\omega_S^{(0)}=0$
if $R^{(m)}(i)$ is *Medium* then $f_{FCWM}^{(0)}(R^{(m)}(i))$ is $\omega_M^{(0)}=0$
if $R^{(m)}(i)$ is *Large* then $f_{FCWM}^{(0)}(R^{(m)}(i))$ is $\omega_L^{(0)}=1$

- In the case of $x^{(m)}(i)=1$

if $R^{(m)}(i)$ is *Small* then $f_{FCWM}^{(1)}(R^{(m)}(i))$ is $\omega_S^{(1)}=0$
if $R^{(m)}(i)$ is *Medium* then $f_{FCWM}^{(1)}(R^{(m)}(i))$ is $\omega_M^{(1)}=1$
if $R^{(m)}(i)$ is *Large* then $f_{FCWM}^{(1)}(R^{(m)}(i))$ is $\omega_L^{(1)}=1$

Fuzzy sets of *Small, Medium* and *Large* are shown in Fig.1

In order to give a design method of the FCWM-1 filter, we define each fuzzy set A_p as a triangle shown as Fig.2, membership function is given by

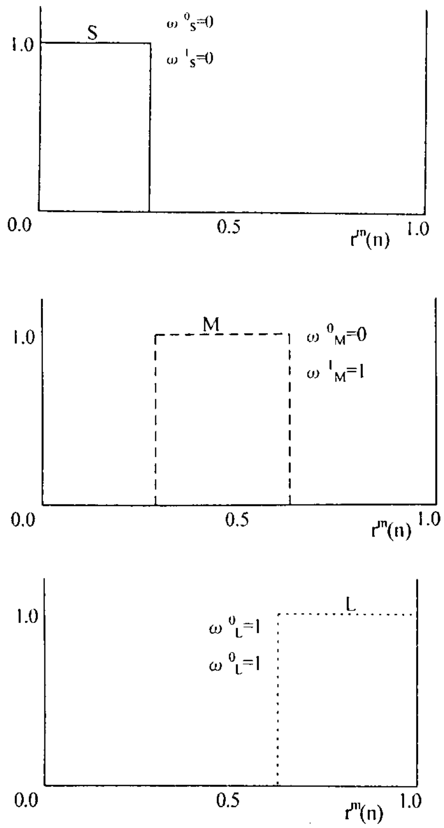

Fig. 1 The CWM filter

$$m_{A_p}(R^{(m)}(i)) = \begin{cases} 1-(a_p - R^{(m)}(i))/b_p^l : a_p - b_p^l < R^{(m)}(i) \le a_p \\ 1-(R^{(m)}(i) - a_p)/b_p^r : a_p < R^{(m)}(i) < a_p + b_p^r \\ 0 \quad : otherwise \end{cases} \tag{10}$$

Stacking property holds, if

$$(a_1 - b_1^l) \le \cdots \le (a_P - b_P^l),\ (a_1 + b_1^r) \le \cdots \le (a_P + b_P^r)$$

and

$$\omega_1^{(0)} \le \cdots \le \omega_P^{(0)},\ \omega_1^{(1)} \le \cdots \le \omega_P^{(1)}$$

are satisfied.

In this case, fuzzy sets and output singletons $\omega_p^{(0)}$, $\omega_p^{(1)}$ are tuned by LMS algorithm. An error function to be minimized is given as follows:

$$J(i) = \frac{1}{2}[y(i) - s(i)]^2 \tag{11}$$

where $y(i)$ is the output of FCWM filter and $s(i)$ is the original signal of training signal. Then, fuzzy sets are able to be tuned by following formula, which gives parameters a_p, b_p^l, b_p^r (see Fig. 1) adaptation:

$$\begin{aligned} a_p(i) &= a_p(i-1) - \gamma_a(\partial J(i)/\partial a_p) \\ &= a_p(i-1) - \gamma_a(y(i) - s(i)) \\ &\quad \times \sum_{m=1}^{M-1} \frac{\{\omega_p^{(+)}(i) - y^{(m)}(i)\} \cdot sgn\{R^{(m)}(i) - a_p(i-1)\}}{b_p^*(i-1) \cdot \sum_{p=1}^{P} m_{A_p}\{R^{(m)}(i)\}} \end{aligned} \tag{12}$$

$$\begin{aligned} b_p^*(i) &= b_p^*(i-1) - \gamma_b(\partial J(i)/\partial b_p^*) \\ &= b_p^*(i-1) - \gamma_b \cdot (y(i) - s(i)) \\ &\quad \times \sum_{m=1}^{M-1} \frac{\{\omega_p^{(+)} - y^m(i)\} \cdot [1 - m_{A_p}\{R^{(m)}(i)\}]}{b_p^*(i-1) \cdot \sum_{p=1}^{P} m_{A_p}\{R^{(m)}(i)\}} \end{aligned} \tag{13}$$

Similarly, ω_p is tuned by

$$\begin{aligned} \omega_p^{(+)}(i) &= \omega_p^{(+)}(i-1) - \gamma_\omega(\partial J(i)/\partial \omega_p^{(+)}) \\ &= \omega_p^{(+)}(i-1) - \gamma_\omega(y(i) - s(i)) \\ &\quad \times \sum_{m=1}^{M-1} \frac{m_{A_p}\{R^{(m)}(i)\}}{\sum_{p=1}^{P} m_{A_p}\{R^{(m)}(i)\}} \end{aligned} \tag{14}$$

2.2.2 Fuzzy Boolean function is directly approximated by the sigmoidal function: FCWM-2

We attempt to approximate the fuzzy Boolean function by the sigmoidal function. Thus, fuzzy Boolean functions of the FCWM-2 filter are given by

$$\begin{aligned} f_{FCWM}^{(0)}(R^{(m)}(i)) &= 1/[1 + \exp\{-\alpha_0(R^{(m)}(i) - \beta_0)\}] \\ f_{FCWM}^{(1)}(R^{(m)}(i)) &= 1/[1 + \exp\{-\alpha_1(R^{(m)}(i) - \beta_1)\}] \end{aligned} \tag{15}$$

Comparing eq.(7) and eq.(15), if α_0 and α_1 are set as large value, β_0 and β_1 are set as adequately, the FCWM-2 filter is equivalent to the CWM filter.

We can also derive the optimal fuzzy Boolean functions following adaptive LMS algorithm

$$\begin{aligned} \alpha_+(i) &= \alpha_+(i-1) - \gamma_\alpha(\partial J(i)/\partial \alpha_+) \\ &= \alpha_+(i-1) - \gamma_\alpha(y(n) - s(n)) \\ &\quad \times \sum_{m=1}^{M-1} \frac{(R^{(m)}(i) - \beta_+(i-1)\exp[-\alpha_+(i-1)\{R^{(m)}(i) - \beta(i-1)\}]}{(1 + \exp[-\alpha_+(i-1)\{R^{(m)}(i) - \beta(i-1)])^2} \end{aligned} \tag{16}$$

$$\beta_+(i) = \beta_+(i-1) - \gamma_\alpha(\partial J(i)/\partial\beta_+)$$
$$= \beta_+(i-1) - \gamma_\alpha(y(i) - s(i))$$
$$\times \sum_{m=1}^{M-1} \frac{-\alpha_+(i-1)\exp[-\alpha_+(i-1)\{R^{(m)}(i) - \beta_+(i-1)\}]}{(1+\exp[-\alpha(i-1)\{R^{(m)}(i) - \beta_+(i-1)\}])^2} \quad (17)$$

where $J(i)$ is an error function which is defined by equation (11).

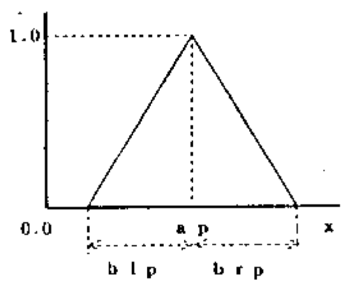

Fig.2 Fuzzy set for FCWM-1

3 EXPERIMENTAL RESULTS

In the experiment, we use the image "Girl"of size 256X256. We prepare 3 noisy images (original image+ Gaussian noise (zero mean and variance 200) + [1%, 1%] or [2%, 2%] or [4%,4%] impulse noise). [a%,b%] means a% positive impulse and b% negative impulse are mixed. The lower right quarter image which is corrupted by zero mean Gaussian noise with variance 200 plus [2%,2%] impulse was used to synthesis the optimal two FCWM filters with 3X3 window. The FCWM-1 filter is constructed by 5 fuzzy rules. Figure 3 shows fuzzy Boolean functions for FCWM-1 and FCWM-2 filters. The fuzzy Boolean function of FCWM-1 show more gentle slope than that of the FCWM-2. Thus, Impulse removing ability of FCWM-2 is superior to that of FCWM-1. On the other hand, Gaussian noise removing ability of FCWM-1 is superior to that of FCWM-2.

Experimental results are shown in Table 1. It is clear that FCWM filters are superior to conventional CWM filter.

Table 1 MSE results

	FCWM-1	FCWM-2	CWM
[1%,1%]	56.54	57.42	68.37
[2%,2%]	62.75	61.59	71.43
[4%,4%]	81.39	76.35	80.62

4 CONCLUSIONS

In this paper, the simplest fuzzy stack filter named fuzzy center weighted median filters are proposed. Two design methods for FCWM filters under the MSE criterion are given by using LMS algorithm. Non-impulse noise removing ability of proposed filters are superior to CWM filters.

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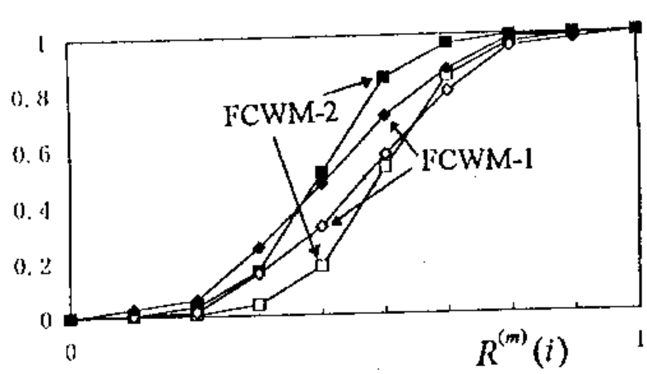

$R^{(m)}(i)$

Fig.3
Fuzzy Boolean function