LATTICE STRUCTURE FOR TWO-CHANNEL FILTER BANKS WITH COMPLEX COEFFICIENTS, WHICH YIELD SYMMETRIC WAVELET BASES

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ABSTRACT

A new lattice structure is described. It is capable of implementing all paraunitary two-channel filter banks where the filters have complex coefficients and yield symmetric wavelet bases. This lattice structure, while being a general design method, can also be used to actually design the filter bank. These filter banks are, in fact, a special case of multifilter banks and can also be related to Golay-Rudin-Shapiro complementary polynomial pairs. The applications of such filter banks are to be found in subband coding and communications systems.

1 INTRODUCTION

The first perfect-reconstruction filter banks were constructed by several researchers independently around 1984 (see the appropriate references in [1]). Later it was recognized that these filter banks are paraunitary and provide orthonormal bases for the Hilbert space. The lattice structure found by Vaidyanathan [2, 3] is an efficient way to implement these filter banks. It has two important properties: (i) without sacrificing computational efficiency it preserves the perfect-reconstruction property even under the constraints of finite-word-length arithmetic, and (ii) it is general, every paraunitary filter bank can be implemented using the lattice structure. The generality of the lattice structure suggests that it can also be used to design the filter bank. However, the three properties: (i) symmetric scaling functions and wavelets, (ii) orthonormality, and (iii) real coefficients can not be achieved simultaneously in the design of orthogonal two-channel filter banks with real coefficients. We must give up one of them. Two-channel orthogonal complex-coefficient filter banks have been recently designed by Lawton [6]. Contrary to what is claimed the filters do not have linear phase. Nevertheless the scaling functions and wavelets are symmetric. In the paper we present, for the first time, lattice structures for filter banks with complex coefficients. These lattice structures have all the advantages of lattice structures i. e. all complex coefficient filter banks that yield symmetric limit functions can be obtained as special cases. The modulation and polyphase matrices of paraunitary filter banks are paraunitary. This leads to the properties

$$H_1(z) = c\, z^{-L}\, \tilde{H}_0(-z) \tag{1}$$

and

$$H_0(z)\tilde{H}_0(z) + |c|^2 \tilde{H}_0(-z)\, H_0(-z) = 2d \tag{2}$$

where $|c|^2 = 1$ and d is an arbitrary constant. The known paraunitary lattice [2, 3] is based on the matrix

$$\boldsymbol{R}_m = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix}, \tag{3}$$

The polyphase matrix of every paraunitary filter bank can be factored in the following way $\mathrm{H}_p(z) = \alpha \boldsymbol{R}_N\, \Lambda(z)\, \boldsymbol{R}_{N-1}\, \Lambda(z) \cdots \boldsymbol{R}_0 \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

2 PARAUNITARY FILTER BANKS WITH COMPLEX COEFFICIENTS

Lawton's construction starts with the lowpass filter $H_0(z)$ in a paraunitary filter bank with real coefficients and replaces every zero λ_k^* with λ_k^{-1} [6]. The orthogonal complex-coefficient filter is given by

$$H_0(z) = h_0(0)(1+z^{-1})^N/2$$
$$\prod_{k=1}^{M} \left[\lambda_k (1-\lambda_k z^{-1})(1-\lambda_k^{-1} z^{-1})\right]. \tag{4}$$

The advantage of doing this is that symmetry is possible when there are no zeros on the real axis, with the exception of zeros at $z = -1$. This is the problem – it is difficult to determine *apriori* whether the approximation technique would result in zeros on the real axis. As a result the design is based on trial and error. If there are zeros on the real axis we can still obtain a complex-coefficient filter, but the main advantage – symmetry – will be lost. The real and imaginary parts are themselves linear-phase, but the complex filter is *not* linear phase.

2.1 The new lattice

Here we are looking for a lattice, based on a building block of the type

$$\boldsymbol{R}_m = \begin{pmatrix} 1 & s_m \\ s_m & 1 \end{pmatrix}. \tag{5}$$

where the coefficients are allowed to be complex. Clearly paraunitaryness, linear phase and real coefficients cannot be achieved simultaneously. (They can be achieved only if the filter coefficients become matrices). The building block must be paraunitary

$$\tilde{\boldsymbol{R}}_m \, \boldsymbol{R}_m = c\boldsymbol{I} \tag{6}$$

which leads to

$$s_m + s_m^* = 0. \tag{7}$$

Therefore $s_m = j\ r_m$, where r_m is a real number. The polyphase matrix of every paraunitary filter bank which yields symmetric limit functions can be factored as follows:

$$\boldsymbol{H}_p(z) = \boldsymbol{T}\ \boldsymbol{\Lambda}(z)\ \boldsymbol{R}_J\ \boldsymbol{\Lambda}(z)\cdots\boldsymbol{R}_1\ . \tag{8}$$

$$\boldsymbol{T} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \tag{9}$$

are simple paraunitary building blocks.

It must be noted that for real-coefficient filter banks the wavelet basis is symmetric if and only if the filters have linear phase. This is not valid for complex-coefficient filter banks. The wavelet basis functions can be symmetric if the Re and Im parts are linear phase. The filters satisfy the propertis: (i) $h_1(n) = (-1)^n\ h_0^*(N-1-n)$ and (ii) $h_0(n) = h_0(N-1-n)$, $h_1(n) = -h_1(N-1-n)$. The filters have symmetric frequency responses with respect to the origin. we have

$$P_m(z) = P_{m-1}(z) + j\ r_m z^{-2}\ Q_{m-1}(z) \tag{10}$$

$$Q_m(z) = jr_m\ P_{m-1}(z) + z^{-2}\ Q_{m-1}(z) \tag{11}$$

and

$$Q_{m-1}(z) = z^{-(2m-3)}\ P_{m-1}(z^{-1}) \tag{12}$$

it can be proven that

$$Q_m(z) = z^{-(2m-1)}\ P_m(z^{-1})\ . \tag{13}$$

By recursion it was established that

$$Q_J(z) = z^{-(2J-1)} P_J(z^{-1})\ . \tag{14}$$

Finally for the transfer functions of the filters

$$\begin{pmatrix} H_0(z) \\ H_1(z) \end{pmatrix} = \begin{pmatrix} h(0) & 0 \\ 0 & h^*(0) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} P_J(z) \\ z^{-2}Q_J(z) \end{pmatrix} \tag{15}$$

it is obtained

$$H_0(z) = h(0)\left[P_J(z) + z^{-2}Q_J(z)\right] \tag{16}$$

$$H_1(z) = h^*(0)\left[P_J(z) - z^{-2}Q_J(z)\right] \tag{17}$$

These equations can be used to find the impulse response coefficients from the lattice coefficients, and vice-versa, to find the lattice coefficients from the impulse response. From (14) it follows that $H_0(z) = z^{-(N-1)}H_0(z^{-1})$ and $H_1(z) = z^{-(N-1)}H_1(z^{-1})$. Note that the lattice is computationally very efficient. If the filters have N taps, then there are $N/2-1$ lattice blocks, each requiring 4 real multiplications and 4 additions. There is a final stage with 2 complex multiplications. If the polyphase components are implemented directly then N complex multiplications and additions would be necessary. Asymptotically the computational savings are on the order of 50 percent. The savings are considerable even for filters of short order.

2.2 Design of the lattice

The filter bank can be designed by optimizing the lattice coefficients directly so that stop-band energy of the lowpass filter

$$E_s = \int_{\omega_s}^{\pi} |H_0(e^{j\omega})|^2\ d\omega \tag{18}$$

is minimum. First the impulse response is found from the lattice coefficients. $H_0^2(e^{j\omega})$ is the frequency response of the half-band filter with positive frequency response, and numerical integration can avoided. To provide starting values which are close to the global optimum the number of lattice stages in increased gradually by adding two lattice stages at once. The hierarchical property is observed, where the frequency responses get better and better with increasing the number of lattice stages.

2.3 Examples

When $J = 2$ we have

$$\boldsymbol{H}_p(z) = \boldsymbol{T}\ \boldsymbol{\Lambda}(z)\boldsymbol{R}_2\ \boldsymbol{\Lambda}(z)\boldsymbol{R}_1 \tag{19}$$

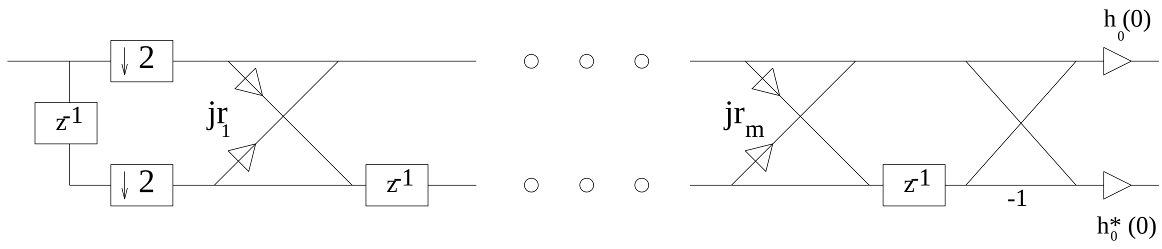

Figure 1: Fig. 1 The paraunitary lattice

The filters have transfer functions

$$H_0(z) = 1 + j\, r_1\, z^{-1} - (r_2\, r_1 - j\, r_2) z^{-2}$$
$$- \quad (r_2\, r_1 - j\, r_2)\, z^{-3} + j\, r_1\, z^{-4} + z^{-5} \qquad (20)$$
$$H_1(z) = 1 + j\, r_1\, z^{-1} - (r_2\, r_1 + j\, r_2) z^{-2}$$
$$+ \quad (r_2\, r_1 + j\, r_2)\, z^{-3} - j\, r_1\, z^{-4} - z^{-5} \qquad (21)$$

An important special case are the filters with maximum number of vanishing moments, considered in [6]. They can be implemented using the lattice coefficients $r_1 = 1.290973$ and $r_2 = 3.872919$. If, however, the filters are designed so that the stopband energy is minimum using the outlined method then the following values for the lattice coefficients are obtained: $r_1 = 0.849458$ and $r_2 = 1.85203$.

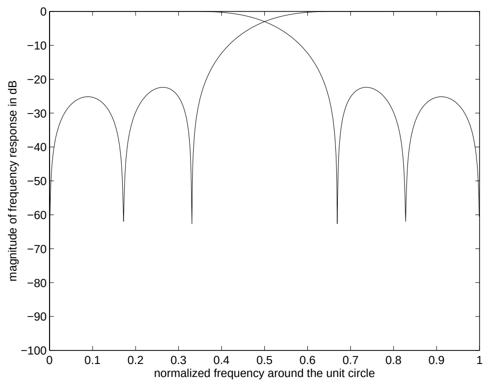

Fig. 2 Frequency responses of a paraunitary pair with 10 taps

3 RELATIONSHIPS WITH MULTIWAVELETS

Multiwavelets are a recent addition to the wavelet theory [7]. The coefficients of multifilters are matrices. Since commutativity in general does not hold, the theory of multifilters is not a straightforward extension of the theory of scalar filter banks. The advantage of using multifilters is that orthogonality and symmetry are possible in the two-channel case. Multifilters operate on vector signals and thus can capture the redundancies present among the components of the vectors.

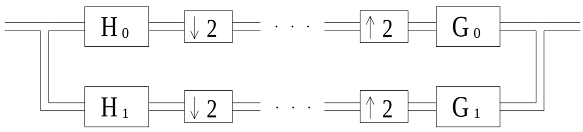

Fig. 3 Two-channel multifilter bank: the signal processing point of view

The complex-coefficient filter banks considered in this paper are, in fact, a special case of multifilters. The two-input two-output multifilters

$$\boldsymbol{H}_0 = \begin{pmatrix} H_0^R(z) & -H_0^I(z) \\ H_0^I(z) & H_0^R(z) \end{pmatrix} \qquad (22)$$

$$\boldsymbol{H}_1 = \begin{pmatrix} H_1^R(z) & -H_1^I(z) \\ H_1^I(z) & H_1^R(z) \end{pmatrix} \qquad (23)$$

are an orthogonal PR multiwavelet filter pair. Since H_i^R and H_i^I are linear-phase, the multifilters $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ have also linear phase. This multifilter bank is commutative, since matrices of the type (22) and (23) commute. The scaling function $\boldsymbol{\phi}(x)$ and the wavelet $\boldsymbol{\psi}(x)$ are vectors. Figs. 4 and 5 illustrate the limit functions that are obtained starting with $\boldsymbol{e} = [11]$ for 6-tap complex filter bank (or multifilter bank) with maximum number of vanishing moments.

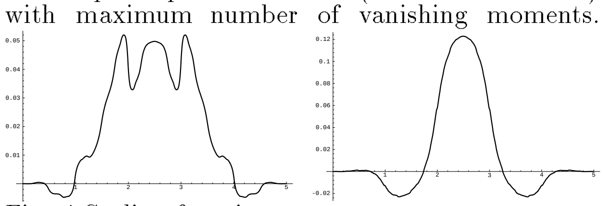

Fig. 4 Scaling functions

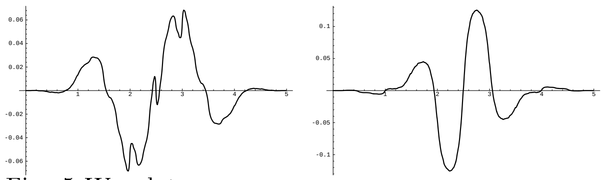

Fig. 5 Wavelets

Our approach is the only existing general design technique for orthogonal and linear phase multiwavelets.

4 COMPLEX TRANSMULTIPLEXERS

Fig. 6 describes the way communications engineers view filter banks. If the filter bank is perfect-reconstruction, then the transmultiplexer is perfect as well. In this way we can theoretically double the number of users if we assign different users to the real and imaginary parts. The filter banks can be arranged in a tree structure to achieve the bandwidth-on-demand property.

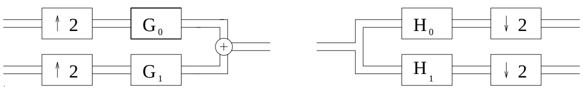

Fig. 6 The communications point of view

5 RELATIONSHIP WITH GOLAY-RUDIN-SHAPIRO POLYNOMIALS

Two polynomials $A(z)$ and $B(z)$, with binary coefficients 1 or -1 are complementary if and only if they satisfy the identity

$$A(z)A(z^{-1}) + B(z)B(z^{-1}) = 2l, \qquad (24)$$

where l is the number of coefficients. It has remained unnoticed the fact that the first perfect-reconstruction filter banks were designed in 1949 by M. Golay:

Proposition 1 *[7] The Golay-Rudin-Shapiro (GRS) polynomial pairs are polyphase components of a lowpass filter in an orthogonal maximally-decimated two-channel FIR filter bank.*

These complementary polynomials provide computationally efficient (no multiplications) orthonormal bases for digital signals. It must be noted that such complementary polynomials are highly *nonregular* filter banks. There are signal processing applications like spread-spectrum communications where regularity is not useful! In the context of this paper we can define complementary polynomials with complex-valued coefficients $\pm 1 \pm j$ as

$$A(z)A^*(z^{-1}) + B(z)B^*(z^{-1}) = \text{const.} \qquad (25)$$

These complementary polynomials are again the polyphase components of a complex lowpass filter in a paraunitary filter bank. The multifilter bank framework can also be used leading to *matrix complementary polynomials* [7].

6 CONCLUSIONS

In this paper a new lattice structure for complex-coefficient filter banks was described. All paraunitary filter banks with complex coefficients that have linear phase real and imaginary parts can be implemented and the lattice can be used as a general design method. The new lattice, in fact, implements one class of orthogonal linear-phase multifilter banks.

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