ON PERFECT-RECONSTRUCTION FIR FILTER BANKS

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ABSTRACT

This paper deals with the problem of designing an N-band maximally-decimated analysis filter bank given K of its filters, so that perfect reconstruction with FIR synthesis filters is possible. An algorithm for computing the $N-K$ unknown analysis filters and the synthesis filters is given and the solution set is completely parametrized. The parametrization is exploited in optimizing the frequency responses of the resulting filters and to derive also a simple parametrization for the paraunitary case. The linear-phase case is also discussed with emphasis on the 2-band filter banks. An example is provided to illustrate the theory.

1 INTRODUCTION

Multirate filter banks find extensive applications in fields like subband coding, transmultiplexer design, design of wavelet bases, etc. [1]. FIR perfect-reconstruction (PR) filter banks (FB's) (i.e., FB's with FIR filters in both the analysis and the synthesis stage) are particularly useful for several reasons, including the possibility of imposing stability and linear phase without sacrificing causality. The problem of determining the analysis filters, given one of them, so that FIR PR is possible, has been addressed in the literature in the context of wavelet design (e.g., [2, 3]) and to facilitate the design and realization of specific classes of PRFB's (e.g., [4]). In [2] the above problem for the case of 2-channel FB's was considered and it was shown to have an analogy with the solution of a polynomial Bezout equation of the form $H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2z^{-(2l+1)}$. The Euclidean algorithm has thus appeared to be a possible approach to solving for a particular complementary filter $\hat{H}_1(z)$, and the solutions were parametrized as

$$H_1(z) = z^{-2k}\hat{H}_1(z) + E(z^2)H_0(z), \qquad (1)$$

where k is any nonnegative integer and $E(z)$ ranges over all polynomials.

This paper deals with the more general version of this problem, where, in general, $1 \leq K \leq N-1$ filters of an N-band maximally-decimated FIR FB are given. It is shown that particular solutions for the unknown filters can be computed via a Smith decomposition of the known part of the analysis polyphase matrix, thus generalizing the approach of [2] to the multiband case. Through this procedure, the synthesis filters are also automatically determined. We provide a complete parametrization of the set of the so-called complementary filters, which in turn yields a characterization of the set of synthesis FB's. Having a complete parametrization of the complementary filters at our disposal allows us to develop optimization procedures for their frequency response characteristics, without worrying about preserving PR.

Linear-phase (LP) FB's are highly desired in several applications, with that of image subband coding being the most well-known. The above problem, referred to hereafter as the (N, K)-problem, is also treated in this paper for the case of LP PRFB's. Since the Euclidean algorithm (viz., Smith reduction) does not preserve the LP property, the parametrization result previously derived is invoked to enforce LP for the complementary filters. The problem is greatly simplified in the special case of $N = 2$, therefore it is discussed in detail. A design example for a 2-band LP PRFB demonstrates the advantages of providing the designer the possibility of fixing one of the filters a-priori.

2 THE (N, K)-PROBLEM

In a PR analysis/synthesis system (ASS), the analysis and synthesis polyphase matrices are related as $\boldsymbol{G}_p^T(z)\boldsymbol{H}_p(z) = \boldsymbol{I}$. Thus, if we require that the synthesis filters be FIR, we must ensure that the matrix $\boldsymbol{H}_p(z)$ is unimodular [1], i.e., $\det \boldsymbol{H}_p(z) = cz^{-k}$. We may thus state the (N, K)-problem as completing the $K \times N$ matrix

$$\boldsymbol{\mathcal{H}}_K(z) = \begin{bmatrix} H_{0,0}(z) & \cdots & H_{0,N-1}(z) \\ \vdots & \ddots & \vdots \\ H_{K-1,0}(z) & \cdots & H_{K-1,N-1}(z) \end{bmatrix} \qquad (2)$$

up to an $N \times N$ unimodular matrix. It is known that this is possible if and only if the matrix (2) is irreducible, that is, its $K \times K$ and $K \times (N-K)$ submatrices are left coprime [5]. This condition can be shown to be generically true, that is, it holds for almost any choice of the first K filters [5]. For the special case of $K=1$, it reduces to the requirement that $H_0(z)$ does not have any factor of the form $z^{-N}-\alpha$, which generalizes the corresponding result of [2].

Via the matrix Euclidean algorithm we can find a unimodular matrix $\boldsymbol{B}(z)$ such that, without loss of generality,

$$\mathcal{H}_K(z)\boldsymbol{B}(z) = \left[\begin{array}{cc} \boldsymbol{I}_K & \boldsymbol{0} \end{array}\right]. \tag{3}$$

Since $\boldsymbol{B}^{-1}(z)$ is polynomial, it follows from (3) that the matrix

$$\hat{\boldsymbol{H}}_p(z) = \left[\begin{array}{c} \mathcal{H}_K(z) \\ \hat{\mathcal{H}}_K(z) \end{array}\right] = \boldsymbol{B}^{-1}(z) \tag{4}$$

is a solution to our problem. The general solution can be parametrized in terms of a particular one as follows:

Theorem 1 *[6] The set of valid analysis polyphase matrices is generated as:*

$$\boldsymbol{H}_p(z) = \left[\begin{array}{cc} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{E}(z) & \boldsymbol{U}(z) \end{array}\right] \hat{\boldsymbol{H}}_p(z), \tag{5}$$

where $\boldsymbol{U}(z)$ is any $(N-K)\times(N-K)$ unimodular matrix and $\boldsymbol{E}(z)$ ranges over all $(N-K)\times K$ polynomial matrices.

Clearly, a solution for the synthesis FB is provided by $\hat{\boldsymbol{G}}_p^T(z) = \boldsymbol{B}(z)$. From (5), the general synthesis polyphase matrix is expressed in terms of a particular one as:

$$\boldsymbol{G}_p(z) = \left[\begin{array}{cc} \boldsymbol{I} & -\boldsymbol{E}^T(z)\boldsymbol{U}^{-T}(z) \\ \boldsymbol{0} & \boldsymbol{U}^{-T}(z) \end{array}\right] \hat{\boldsymbol{G}}_p(z). \tag{6}$$

The $(N,1)$-problem for paraunitary FB's has already been considered [4, 3] but the proposed approaches are restricted to the case that the McMillan degree of $\boldsymbol{H}_p(z)$ is equal to that of its first row. Having determined $\hat{\boldsymbol{H}}_p(z)$ as suggested in [4], simply applying eq. (5) (with $\boldsymbol{E}(z)=\boldsymbol{0}$ because of the paraunitariness condition) we obtain a characterization of the class of paraunitary matrices with fixed first row, which overcomes the degree constraint mentioned above. Now $\boldsymbol{U}(z)$ ranges over all paraunitary matrices of size $N-1$, and in the special case treated in [4, 3], it becomes a constant unitary matrix, which, if real, is parametrizable with $\left(\begin{array}{c} N-1 \\ 2 \end{array}\right)$ parameters, agreeing with [4, 3].

3 THE LINEAR-PHASE CASE

Even if the given filters are LP, the proposed algorithm does not guarantee the LP property for the solution filters, hence the question of how to choose the parameters in Theorem 1 so as to enforce LP arises. Due to lack of space, we will only give the results here, without proof. Details can be found in [6]. The literature on LP multiband FB's has been confined to the class of FB's whose filter lengths M_i are equal modulo N, i.e., $M_i = m_i N + I + 1$, with $0 \le I \le N-1$. The polyphase matrix in such a FB satisfies [7]

$$\boldsymbol{H}_p(z^{-1}) = \boldsymbol{D}_N \boldsymbol{\Lambda}_N(z)\boldsymbol{H}_p(z)\boldsymbol{P}(z),$$

where $\boldsymbol{D}_l = \mathrm{diag}(J_0, \ldots, J_{l-1})$ with $J_i \in \{-1,1\}$ signifying the type of symmetry of $H_i(z)$, $\boldsymbol{\Lambda}_l(z) = \mathrm{diag}(z^{m_0}, \ldots, z^{m_{l-1}})$ and

$$\boldsymbol{P}(z) = \left[\begin{array}{cc} \boldsymbol{J}_{I+1} & \boldsymbol{0} \\ \boldsymbol{0} & z^{-1}\boldsymbol{J}_{N-I-1} \end{array}\right]$$

where $\boldsymbol{J}_n$ denotes the exchange matrix of order n.

Theorem 2 *The solutions to the LP $(N, N-1)$-problem are generated by (5) where $\hat{\boldsymbol{H}}_p(z)$ is found via*

$$k = \frac{1}{2}\left(\sum_{i=0}^{N-1} m_i - N + I + 1\right) \tag{7}$$

and

$$\begin{aligned} J_{N-1} z^{m_{N-1}} \boldsymbol{E}(z)\boldsymbol{D}_{N-1}\boldsymbol{\Lambda}_{N-1}(z^{-1}) - \boldsymbol{E}(z^{-1}) \quad &= \\ -\, J_{N-1} z^{m_{N-1}-k}\boldsymbol{X}(z) \quad & \end{aligned} \tag{8}$$

with $\left[\begin{array}{cc} \boldsymbol{X}(z) & J_{N-1} z^{\sum_{i=0}^{N-2} m_i - N + I + 1} \end{array}\right]$ being the last row of the matrix $\boldsymbol{B}^{-1}(z)\boldsymbol{P}(z)\boldsymbol{B}(z^{-1})$. A closed formula for $\boldsymbol{X}(z)$ is provided in [6].

It is well known that there are only two nontrivial classes of 2-band LP PRFB's [8, 2]: (i) different symmetry for $H_0(z)$, $H_1(z)$ and even lengths differing by an even multiple of 2, and (ii) same (positive) symmetry and odd lengths differing by an odd multiple of 2.

Theorem 3 *The LP complementary filter of a length-M symmetric filter $H_0(z)$ is given by (1) where*

(i) $k = \frac{M}{2} - 1$ and

$$\begin{aligned} E(z) + E(z^{-1}) &= [\det \hat{\boldsymbol{H}}_p(z)]^{-1} \times \\ & [\hat{H}_{1,0}(z)\hat{H}_{1,0}(z^{-1}) - \hat{H}_{1,1}(z)\hat{H}_{1,1}(z^{-1})] \end{aligned} \tag{9}$$

if $H_1(z)$ is symmetric of length M, and

(ii) $k = \frac{M-1}{2}$ and

$$\begin{aligned} E(z) - z^{-1}E(z^{-1}) &= [\det \hat{\boldsymbol{H}}_p(z)]^{-1} \times \\ & [z^{-1}\hat{H}_{1,1}(z)\hat{H}_{1,0}(z^{-1}) - \hat{H}_{1,1}(z^{-1})\hat{H}_{1,0}(z)] \end{aligned} \tag{10}$$

if $H_1(z)$ is antisymmetric of length $M+2$.

4 DESIGN CONSIDERATIONS

The problem of optimizing the frequency characteristics of the $N-K$ complementary filters is one of nonlinearly constrained optimization in its most general form, since the unimodularity of $\boldsymbol{U}(z)$ in (5) has to be assured. However, in the case of only one complementary filter ($K = N-1$) the matrix $\boldsymbol{U}(z)$ reduces to a scalar and the optimization can be done only with respect to the elements of $\boldsymbol{E}(z)$, thus removing the nonlinear constraint. We shall adopt the stopband energy, $\mathcal{E} = \int_S |H_{N-1}(e^{j\omega})|^2 d\omega$, as the cost function in the sequel. Letting $\boldsymbol{h}_0(z) = [H_0(z), \ldots, H_{N-2}(z)]^T$ denote the vector of known filters, eq. (5) yields:

$$H_{N-1}(e^{j\omega}) = e^{-jkN\omega}\hat{H}_{N-1}(e^{j\omega}) + \boldsymbol{E}(e^{jN\omega})\boldsymbol{h}_0(e^{j\omega}). \tag{11}$$

Expressing each element of $\boldsymbol{E}(e^{j\omega})$ as $E_i(e^{j\omega}) = \boldsymbol{\epsilon}_i^T \boldsymbol{e}(\omega)$ where $\boldsymbol{e}(\omega) = [1, e^{-j\omega}, \ldots, e^{-j(m-1)\omega}]^T$, and assuming real coefficients, we obtain

$$\mathcal{E} = \hat{\mathcal{E}} + 2\boldsymbol{b}^T\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^T\boldsymbol{P}\boldsymbol{\epsilon}, \tag{12}$$

where $\boldsymbol{\epsilon} = [\boldsymbol{\epsilon}_0^T, \ldots, \boldsymbol{\epsilon}_{N-2}^T]^T$,

$$\hat{\mathcal{E}} = \int_S |\hat{H}_{N-1}(e^{j\omega})|^2 d\omega, \tag{13}$$

$$\boldsymbol{b} = \Re\{\int_S e^{-jkN\omega}\hat{H}_{N-1}(e^{j\omega})\boldsymbol{p}(-\omega)d\omega\}, \tag{14}$$

$$\boldsymbol{P} = \int_S \boldsymbol{p}(\omega)\boldsymbol{p}^H(\omega)d\omega, \tag{15}$$

with $\boldsymbol{p}(\omega) = \boldsymbol{h}_0(e^{j\omega}) \otimes \boldsymbol{e}(N\omega)$, leading to the linear system of equations

$$\boldsymbol{P}\boldsymbol{\epsilon} = -\boldsymbol{b} \tag{16}$$

for the optimum $\boldsymbol{E}(z)$. It can be verified that the $(N-1)m \times (N-1)m$ matrix $\boldsymbol{P}$ is Hermitian, with Toeplitz blocks. Moreover, if the given filters are of high quality, $\boldsymbol{P}$ is approximately block diagonal and positive definite. The rich structure of $\boldsymbol{P}$ allows the use of existing efficient algorithms for the solution of (16).

Example: To demonstrate the potential advantages of the design approach implied by the above theory, we consider an example of a 2-band LP FB with the filters $H_0(z)$, $H_1(z)$ being both symmetric of length 23 and 25, respectively (class (ii)). The magnitude responses of the filters as designed in Example 4.1 of [8] are shown in Fig. 1 (labeled a, d). With $H_0(z)$ being chosen as the filter b of Fig. 1, a particular nonlinear-phase complementary filter $\hat{H}_1(z)$ of length 23 is computed (curve c). Using Theorem 3(ii) and the optimization procedure developed above, the filter e of Fig. 1 results as the optimum LP solution for $H_1(z)$. Clearly, the new filters have higher filtering performance than those designed in [8] by optimizing some of the parameters of a cascade-lattice structure and arbitrarily setting the rest of them. Our design method gives rise to a factorization for $\boldsymbol{H}_p(z)$, of the form:

$$\boldsymbol{H}_p(z) = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ E(z) & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ z^{-11} & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} 1 & 0 \\ q_{10}(z) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & q_1(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q_0(z) & 1 \end{bmatrix} \tag{17}$$

where $q_i(z)$ are first-order polynomials and $E(z)$ is a polynomial of order 10. The above factorization implies a ladder-based realization [9, 10] of the FB. Furthermore, we write the normalizing scalors as:

$$\begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \beta & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix} \times$$
$$\begin{bmatrix} 1 & 1/\beta_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\beta_1 & 1 \end{bmatrix} \tag{18}$$

where $\beta = \beta_1\beta_2$. The effects of quantizing the coefficients in both FB's to 24-bit mantissa and 8-bit exponent are shown in Figs. 2, 3 where the relative deviations of the magnitude responses are plotted. In this precision, both FB's retain the LP property. The preservation of PR under multiplication roundoff in the ladder structure [11] has been verified by computing the error in the reconstruction of a ramp signal (Fig. 4). Counting the computational complexity in terms of the number of multiply-accumulate (MAC) instructions required, we can see that the ladder structure needs 37 MAC as opposed to the 57 MAC per unit time required by the lattice ASS.

5 CONCLUSIONS

The problem of designing an FIR PR ASS given some of the filters of the analysis FB was given a general solution. The parametrization of the solution filters was exploited in optimizing their stopband characteristics and also in obtaining LP solutions. It was shown through an example that our approach compares favorably to earlier designs with respect to complexity, filtering performance and numerical behavior.

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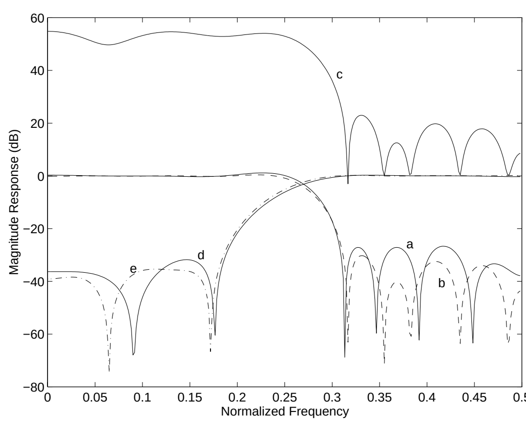

Figure 1: Magnitude responses of the filters in the example FB.

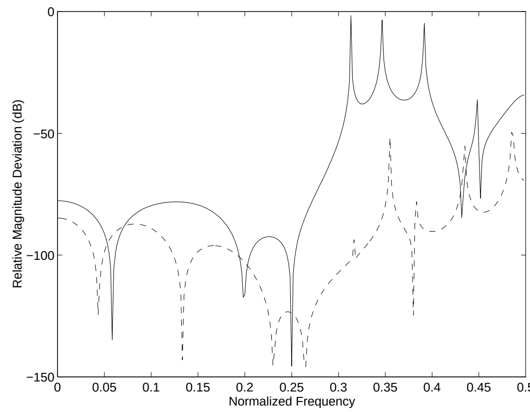

Figure 2: Low-pass filter magnitude deviation (lattice: solid, ladder: dashed).

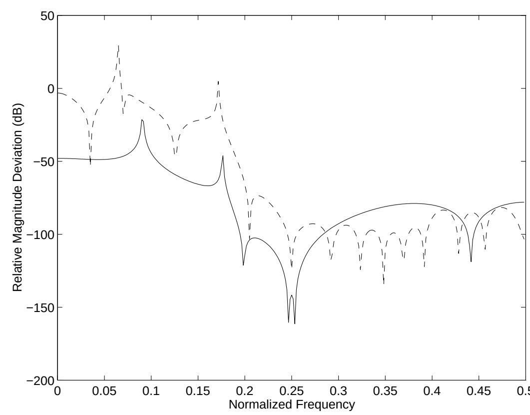

Figure 3: High-pass filter magnitude deviation (lattice: solid, ladder: dashed).

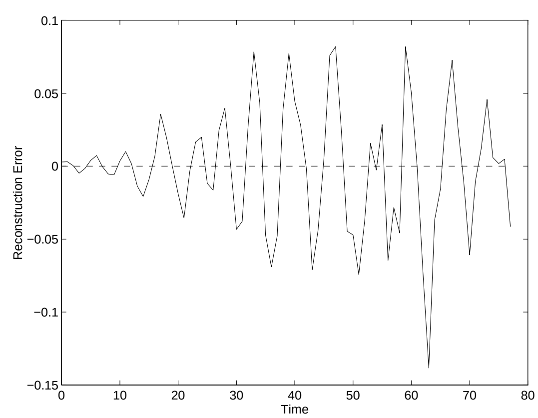

Figure 4: Reconstruction error of the lattice (solid line) and ladder (dashed line) structures.