MEASUREMENT AND SYMBOLIC ANALYSIS OF IMPLEMENTED MULTIRATE SYSTEMS

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ABSTRACT

Multirate systems (MRS) play a major role in modern telecommunication. Important examples are filter banks for image or speech coding, transmultiplexers, and sampling rate converters. In general, these systems are designed without consideration of implementation aspects such as wordlength limitations. The performance of realized systems will therefore differ from the desired one depending on the system structure. Not all deviations can be calculated in closed form and even practicable calculations are often extensive and error-prone. Therefore, we present a method for measuring quantization effects in realized MRS. Furthermore, we introduce a new program for the symbolic analysis of MRS using a computer algebra program.

1 INTRODUCTION

In realized MRS many different types of deviations from the desired behaviour can be observed. First of all, the finite wordlength representation of filter coefficients results in linear distortions of frequency responses and thus gives rise to undesired aliasing components. Furthermore, nonlinear distortion by data quantization yields noise-like components depending on the type of quantization. An important application of multirate technology are lossy subband and transform coders. The coding stage between the analysis and synthesis part of these systems results in additional linear distortions and signal dependent noise-like contributions. Furthermore, transmission channels cause intersymbol interference and also noise-like distortions.

Not all of these deviations from the ideal case can be investigated theoretically. Furthermore, the complexity of today's signal processing systems makes a closed form calculation often very difficult, extensive, and extremely error-prone. Therefore, we first introduce a method for measuring the performance of implemented MRS without prior knowledge of the system structure. In addition we present a new program for an automated closed form analysis of MRS using the computer algebra program Maple. Here we are going to focus on linear distortions and the measurement and closed form analysis of these deviations.

This paper is based on a matrix description of linear MRS [4] that is well suited for a description of arbitrary MRS and an easy analysis of compound systems.

2 DESCRIPTION OF LINEAR MRS

In the following, k_1 and k_2 are discrete time variables at the input and output of an MRS with in general different sampling rates. Their z-domain variables are denoted by z_1 and z_2. L_1 and L_2 are blocking factors of an equivalent block processing system (see fig. 1). Furthermore, we refer to a common sampling rate from which all sampling rates in the MRS can be derived by downsampling. With the corresponding z-domain variable z_0 we get $z_1 = z_0^{\ell_2}$ and $z_2 = z_0^{\ell_1}$ with $\ell_\nu = L_\nu/n$ and $n = \gcd\{L_1, L_2\}$.

2.1 Polyphase (PP) Description

Each MRS can be described by an equivalent block processing system [7, 4]. Using the type 1 PP decomposition [7] of the input signal $v(k_1)$ and of the output signal $y(k_2)$ it can be shown [4] that the PP vectors

$$\mathbf{v}^{(\mathrm{p}:L_1)}(\kappa) = \left[\, v(L_1\kappa)\ v(L_1\kappa+1)\ \ldots\ v(L_1\kappa+L_1-1)\,\right]^{\mathrm{T}}$$
$$\mathbf{y}^{(\mathrm{p}:L_2)}(\kappa) = \left[\, y(L_2\kappa)\ y(L_2\kappa+1)\ \ldots\ y(L_2\kappa+L_2-1)\,\right]^{\mathrm{T}}$$

are related by $\mathbf{y}^{(\mathrm{p}:L_2)}(\kappa) = \mathbf{h}^{(\mathrm{p}:L_2\times L_1)}(\kappa) * \mathbf{v}^{(\mathrm{p}:L_1)}(\kappa)$, where the time-domain PP matrix $\mathbf{h}^{(\mathrm{p}:L_2\times L_1)}(\kappa)$ describes a linear time-invariant (LTI) multi-input multi-output (MIMO) system. The relation can be formulated in the z-domain as well

$$\mathbf{Y}^{(\mathrm{p}:L_2)}(z) = \mathbf{H}^{(\mathrm{p}:L_2\times L_1)}(z) \cdot \mathbf{V}^{(\mathrm{p}:L_1)}(z)$$

using the z-transform with respect to κ. An MRS may thus be represented equivalently as a cascade of a PP analysis network (PPAN), an LTI MIMO system, and a PP synthesis network (PPSN), as shown in fig. 1.

2.2 Modulation Description

Another important form of describing MRS is the modulation or alias-component representation [4]. Using the

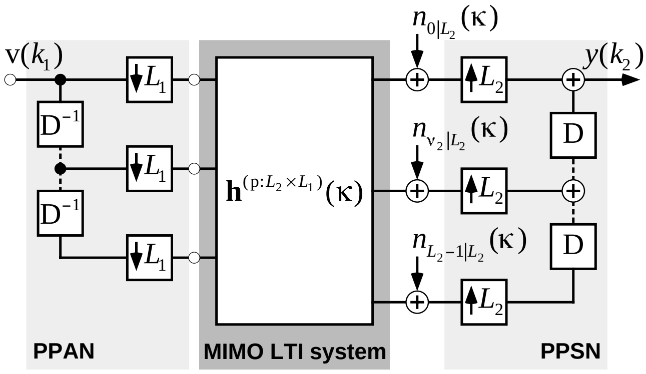

Figure 1: Equivalent representation of MRS

z-domain modulation vector

$$\mathbf{V}^{(\mathrm{m}:L_1)}(z_1) = \frac{1}{\sqrt{L_1}}\left[V(z_1)\; V(w_{L_1}^{-1}z_1)\;\ldots\; V(w_{L_1}^{-(L_1-1)}z_1)\right]^{\mathrm{T}}$$

of a signal $v(k_1)$ it can be shown that the modulation vectors of the input and output signal of an MRS are related by

$$\mathbf{Y}^{(\mathrm{m}:L_2)}(z_0^{\ell_1}) = \mathbf{H}^{(\mathrm{m}:L_2\times L_1)}(z_0)\cdot\mathbf{V}^{(\mathrm{m}:L_1)}(z_0^{\ell_2}).$$

The modulation matrix

$$\mathbf{H}^{(\mathrm{m}:L_2\times L_1)}(z_0) = \mathbf{W}_{L_2}\mathbf{D}_{L_2}(z_0^{\ell_1})\mathbf{H}^{(\mathrm{p}:L_2\times L_1)}(z_0^{n\ell_1\ell_2})\mathbf{D}_{L_1}^{-1}(z_0^{\ell_2})\mathbf{W}_{L_2}^{-1}$$

of an MRS provides a completely equivalent description of such systems. Here, $\mathbf{W}_M$ denotes the unitary DFT matrix and $\mathbf{D}_M(z) = \mathrm{diag}\left\{z^{-0}, z^{-1}, \ldots, z^{-(M-1)}\right\}$ is a matrix of delay elements.

3 MEASUREMENT

The measurement is based on modelling an implemented MRS as a linear MRS with additive noise sources $n_{\nu_2|L_2}(\kappa)$ (see fig. 1). These noise sources are not necessarily white but are assumed to be orthogonal to the outputs $y_{\mathrm{L},\nu_2|L_2}(\kappa)$ of the linear system, i.e.

$$\mathcal{E}\left\{y_{\mathrm{L},\nu_2|L_2}(\kappa)\cdot n_{\nu_2|L_2}(\kappa)\right\} = 0,\; \nu_2 = 0\ldots L_2-1.$$

Since we are interested in individual transfer functions

$$H_{\nu_2\nu_1}(e^{j\Omega_\mu}) = \left[\mathbf{H}^{(\mathrm{p}:L_2\times L_1)}(e^{j\Omega_\mu})\right]_{\nu_2\nu_1}$$

at discrete frequencies $\Omega_\mu = \frac{2\pi\mu}{M},\; \mu = 0\ldots M-1$ we have to separate several disturbing signal components from the output signal of the system. First of all, we have to remove or at least reduce noise-like distortions caused e.g. by wordlength effects or coding. Furthermore, the output PP components of the linear MRS

$$y_{\mathrm{L},\nu_2|L_2}(\kappa) = \sum_{\nu_1=0}^{L_1-1} h_{\nu_2\nu_1}(\kappa) * v_{\nu_1|L_1}(\kappa)$$

are a superposition of the output signals of L_1 partial systems of the equivalent MIMO system. Obviously, the determination of a specific impulse response $h_{\nu_2\varrho_1}(\kappa)$ requires the separation of signal components resulting from the respective input PP component $v_{\varrho_1|L_1}(\kappa)$.

It was shown in [5] for time-invariant systems that the influence of noise can be reduced significantly using an ensemble of L single measurements with pseudo-random input signals $v_\lambda(k)$, where λ denotes the number of the measurement. The input signals can be constructed using a quite arbitrary deterministic signal $v(k)$ with an additional random phase $\varphi_\lambda \in [-\pi,\pi)$, i.e. $v_\lambda(k) = v(k)\cdot e^{j\varphi_\lambda}$. At the output we then have

$$y_\lambda(k) = h(k) * v(k)\cdot e^{j\varphi_\lambda} + n_\lambda(k).$$

As noise and the output of the linear system are orthogonal it is possible to separate linear signal components by averaging

$$\frac{1}{L}\sum_{\lambda=0}^{L-1} y_\lambda(k)\cdot e^{-j\varphi_\lambda} \approx h(k) * v(k).$$

The same principle may be applied in order to reduce noise-like distortions in an MRS, i.e. we have to use input signals $v_\lambda(k_1) = v(k_1)\cdot e^{j\varphi_\lambda}$. The separation of a specific partial transfer function can be achieved by an additional deterministic phase shift of the input PP components. The resulting input PP components are given by

$$v_{\nu_1|L_1,\lambda}(\kappa) = v_{\nu_1|L_1}(\kappa)\cdot e^{j\varphi_\lambda}\cdot w_L^{-\lambda\nu_1},\; \nu_1 = 0\ldots L_1-1.$$

At the output we have

$$y_{\nu_2|L_2,\lambda}(\kappa) = \sum_{\nu_1=0}^{L_1-1} h_{\nu_2\nu_1}(\kappa) * v_{\nu_1|L_1}(\kappa)\cdot e^{j\varphi_\lambda}\cdot w_L^{-\lambda\nu_1} + n_{\nu_2|L_2,\lambda}(\kappa)$$

and the desired signal component can be determined by

$$\frac{1}{L}\sum_{\lambda=0}^{L-1} y_{\nu_2|L_2,\lambda}(\kappa)\cdot e^{-j\varphi_\lambda}\cdot w_L^{\lambda\varrho_1} \approx h_{\nu_2\varrho_1}(\kappa) * v_{\varrho_1|L_1}(\kappa).$$

For periodic PP components, partial transfer functions at discrete frequencies can be determined easily using the DFT of signals when the steady state is reached [2].

4 SYMBOLIC ANALYSIS

Since the closed form analysis of MRS is a very extensive task we developed a toolbox for the symbolic analysis of MRS using the computer algebra program Maple. Our toolbox is based on a structural description of MRS using some fundamental building blocks (see fig. 2) and their interconnections. Modulation matrices and arbitrary transfer functions within compound systems can then be calculated automatically by appropriate matrix operations, performed with the modulation matrices of fundamental blocks. In the following, we will describe the basic elements and operations of our toolbox.

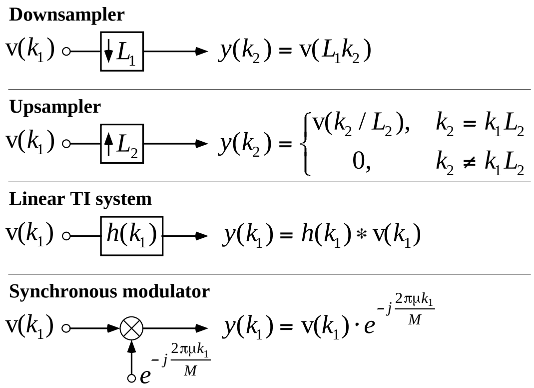

Figure 2: Fundamental building blocks of MRS

4.1 Building Blocks

Downsampler: A downsampler by a factor L_1 is characterized by

$$\begin{aligned}\mathbf{H}_{\text{down}}^{(\text{p}:1\times L_1)}(z) &= \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}\\ \mathbf{H}_{\text{down}}^{(\text{m}:1\times L_1)}(z) &= \frac{1}{\sqrt{L_1}}\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}.\end{aligned}$$

Upsampler: An upsampler by a factor L_2 is characterized by

$$\begin{aligned}\mathbf{H}_{\text{up}}^{(\text{p}:L_2\times 1)}(z) &= \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^{\text{T}}\\ \mathbf{H}_{\text{up}}^{(\text{m}:L_2\times 1)}(z) &= \frac{1}{\sqrt{L_2}}\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^{\text{T}}.\end{aligned}$$

Linear TI System: A linear TI system (digital filter) is characterized by its transfer function $H(z)$ which is identical with its PP and modulation matrix

$$\mathbf{H}_{\text{LTI}}^{(\text{p}:1\times 1)}(z) = \mathbf{H}_{\text{LTI}}^{(\text{m}:1\times 1)}(z) = H(z).$$

Synchronous Modulator: A synchronous modulator with $y(k_1) = v(k_1)\cdot w_M^{\mu k_1}$ is characterized by

$$\begin{aligned}\mathbf{H}_{\text{mod}}^{(\text{p}:M\times M)}(z) &= \text{diag}\left\{ 1 \;\; w_M^{\mu\cdot 1} \;\; \ldots \;\; w_M^{\mu(M-1)} \right\}\\ \mathbf{H}_{\text{mod}}^{(\text{m}:M\times M)}(z) &= \begin{bmatrix} \mathbf{0} & \mathbf{I}_{M-1} \\ 1 & \mathbf{0}^{\text{T}} \end{bmatrix}^{\mu}.\end{aligned}$$

4.2 Required Operations

Expansion of MRS: An LTI system can be represented equivalently [7] as an alias-free $p\times p$ MRS for a simultaneous processing of p subsequent samples. Similarly, an $L_2\times L_1$ MRS can be described as an $pL_2\times pL_1$ MRS. In the course of the analysis of compound MRS this expansion of MRS by an integer factor p often becomes necessary. The modulation matrix of a p-expanded system is given by [4, 6]

$$\mathbf{H}^{(\text{m}:pL_2\times pL_1)}(z) = \sum_{\nu=0}^{p-1}\mathbf{H}^{(\text{m}:L_2\times L_1)}\left(z e^{j\frac{2\pi\nu}{pn\ell_1\ell_2}}\right)\otimes\mathbf{E}_{\nu,\nu}^{p\times p}$$

with $\mathbf{A}\otimes\mathbf{B} = [a_{kl}\mathbf{B}]$ denoting the Kronecker product. All elements of the matrix $\mathbf{E}_{\nu,\nu}^{p\times p}$ are zero except element (ν,ν), which is 1.

Parallel Connection of MRS: A parallel connection of N MRS results in a summation of the corresponding PP or modulation matrices of the same dimension. It is therefore (with $\bullet \mathrel{\hat{=}}$ p or m)

$$\mathbf{H}^{(\bullet:L_2\times L_1)}(z) = \sum_{\nu=1}^{N}\mathbf{H}_{\nu}^{(\bullet:L_2\times L_1)}(z).$$

If the matrices have different dimensions $L_{2\nu}\times L_{1\nu}$ but the same ratio $L_{2\nu}/L_{1\nu}$ the connection can be done after appropriate expansions. If the ratios are different a parallel connection of these systems is not possible.

Serial Connection of MRS: A serial connection of N MRS results in the product of their modulation or PP matrices

$$\mathbf{H}^{(\bullet:L_{2N}\times L_{11})}(\bullet) = \mathbf{H}_N^{(\bullet:L_{2N}\times L_{1N})}(\bullet)\cdot\ldots\cdot\mathbf{H}_1^{(\bullet:L_{21}\times L_{11})}(\bullet).$$

Again, systems expansions are necessary if the matrices do not have proper dimensions for a multiplication. Moreover, the z-domain variable has to refer to a common sampling rate of the overall system.

5 EXAMPLES

5.1 Sampling Rate Converter

As a first simple example we consider an efficient PP implementation of a sampling rate converter (see fig. 3) for changing sampling rates by a fractional factor of 2/3.

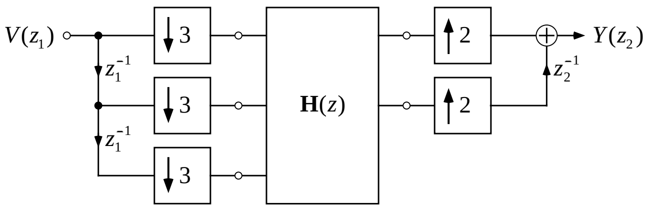

Figure 3: Efficient sampling rate conversion by 2/3

The transfer matrix of the converter is given by

$$\mathbf{H}(z) = \begin{bmatrix} z^{-1}H_{4|6}(z) & H_{0|6}(z) & H_{2|6}(z) \\ H_{1|6}(z) & H_{3|6}(z) & H_{5|6}(z) \end{bmatrix}$$

with the type 1 PP components $H_{\nu|6}(z)$, $\nu = 0\ldots 5$ of an appropriate lowpass filter

$$H(z) = \sum_{\nu=0}^{5} z^{-\nu}H_{\nu|6}(z^6).$$

Using a structural description of the converter, our analysis tool calculates the closed form input-output relation

$$Y(z_0^3) = \frac{1}{3}\sum_{\nu=0}^{5}\sum_{\mu=0}^{2}\left(z_0 e^{j\frac{2\pi\mu}{3}}\right)^{-(\nu+2)} H_{\nu|6}\left(z_0^6\right) V\left(z_0^2 e^{j\frac{4\pi\mu}{3}}\right)$$

automatically. For a visualization we now insert an appropriate linear-phase lowpass filter $H(z)$ of degree 59 with a coefficient wordlength of 8 bits. Then, a comparison reveals an excellent agreement of measurement and symbolic analysis (see fig. 4).

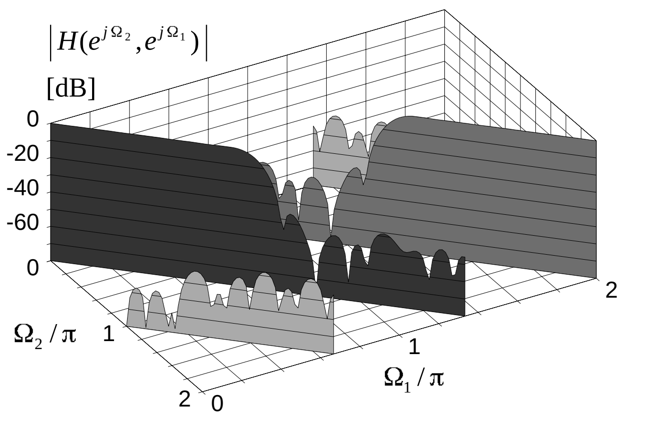

Figure 4: Bifrequency transfer function [1] of the sampling rate converter with quantized coefficients

5.2 2-Channel IIR Filter Bank

Our second example consists of a 2-channel IIR filter bank [7] with power symmetric elliptic filters. It can be realized using the structure in fig. 5 with properly chosen allpass filters $A_0(z)$ and $A_1(z)$ and

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

In the ideal case this system is free from aliasing and am-

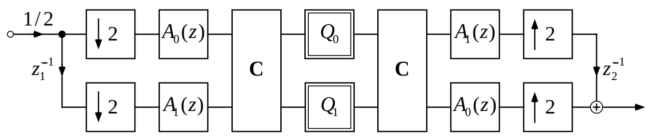

Figure 5: 2-channel IIR filter bank

plitude distortions. Nevertheless, both distortions can be observed if the data are quantized by magnitude truncation. A coding stage with a different quantization step Q_0, Q_1 in both channels yields further distortions (see fig. 6). Since no suitable models of these distortions are known, measuring the system is still necessary for its verification.

6 CONCLUDING REMARKS

We presented a program for the automatic symbolic analysis of MRS using the computer algebra program Maple. A comparison with measurement results confirmed that the symbolic analysis of linear systems yields exact results. The program thus can be employed for the verification and comparison of various system structures and coefficient sets. Nevertheless, our method for measuring implemented MRS is still an indispensable means for verifying those systems and revealing deviations from their expected behaviour. In the future, we intend to de-

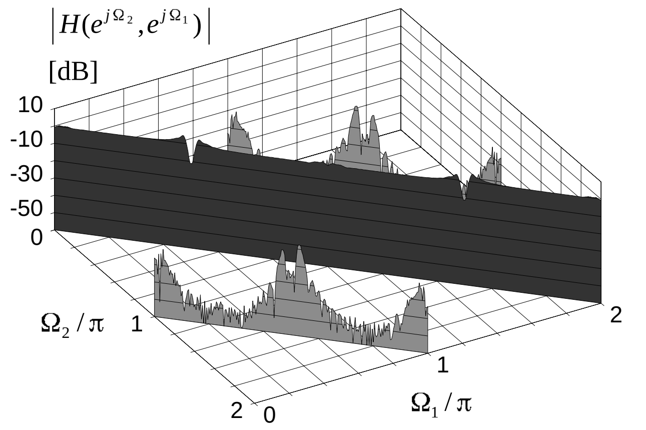

Figure 6: Bifrequency transfer function of the 2-channel IIR filter bank with magnitude truncation and coding

velop appropriate models of wordlength limitations for a further refinement of our analysis tool.

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