SPECTRAL ANALYSIS OF RANDOMLY SAMPLED PROCESSES

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ABSTRACT

The power spectral density of randomly sampled signals is studied with reference to fluid velocity measured by laser Doppler velocimetry.

In this paper, we propose a new method for spectral estimation of Poisson-sampled stochastic processes. Our approach is based on polygonal (first-order) interpolation from the sampled process followed by resampling and usual fast Fourier transform. This study emphasizes the merit of the polygonal hold vs. the sample-and-hold.

1 INTRODUCTION

Conventionally, digital spectral analysis requires the data to be equally spaced. However, it is well known that estimates of power spectra of fluid velocities have been made from data sets which are fondamentally unequally spaced. This situation is generated by the random, intermittent nature of the velocity signal caused by the random arrival of particles at the measurement volume.

Random sampling also occurs in other conditions, either voluntarily to use less memory, reduce processing or reduce bandwidth for digital transmission, or as a consequence of the nature of the measuring system. Although random sampling may be more natural or efficient than uniform sampling, it has not been used for several reasons. First, the nonuniform sampling theory is not simple. Second, the time and frequency analyses are rather involved. Finally, there was no simple and practical reconstruction method for error free recovery as opposed to low pass filtering in the uniform sampling case. It follows that, generally, these data sets are assumed to be equally spaced, or some preprocessing is performed in an attempt to produce an equally spaced data set from the raw, unequally spaced set.

Recently [1],[2], the power spectral density (PSD) of an LDV (Laser Doppler Velocimetry) velocity signal was estimated by sampling at the arrivals of valid signal bursts and holding the values until another valid signal arrives. Unfortunately, the measured spectrum is filtred at the mean rate and it contains a filtered white noise spectrum caused by the steps in the sample-and hold (zero-order interpolator) signal.

Our aim is to reduce the effect of this noise by using a simple first-order (or polygonal) interpolator. The main objectives of the present study can be summarized as follows:

In section 2, the problem of random sampling is formulated, and previously known results [7], [8] are stated. We briefly compare two reconstruction procedures: the zero-order hold and the polygonal hold. The purpose of section 3 is to investigate spectral estimation from Poisson sampling by interpolation and compare the performances of the interpolation schemes in spectral domain for two representative processes encountered in engineering applications and particulary in LDV: the Kolmogorov process and a band-limited Gaussian process. The data for the two schemes are assumed to have the same size (10,000 samples) and equal average sampling rates (10 kHz). In particular, we show how the spectral bias introduced by polygonal interpolation can be compensated.

2 POLYGONAL VS. ZERO-ORDER INTERPOLATORS: MEAN SQUARE COMPARISON

2.1 Hypothesis

Let the physical signal of interest be a stochastic stationary process denoted by $x(t)$. A problem of considerable practical importance is the estimation of the power spectral density $S_x(f)$ of the stochastic process $x(t)$ from a sequence $\{x(t_n)\}$ of its samples delivered by the laser Doppler processor at random time intervals t_n.

The sampling instants $\{t_n\}$ are given by

$$t_n = t_{n-1} + a_n, \quad n = 0, \pm 1, \ldots, \tag{1}$$

where the $\{a_n\}$ are independent identically distributed positive random variables with a common distribution $F(\sigma) = 1 - \exp(-\lambda\sigma)$ (Poisson distribution). It is assumed that the sampling instants $\{t_n\}$ are independent of the process x. Note that 1 is the average sampling rate (where $\lambda = 10$kHz). The sample size is fixed at $N = 10,000$.

Among the sampling schemes which yield theoretically consistent estimates of the power spectral density from the observation $\{x(t_n)\}$, the Poisson sampling scheme is the simplest and the most fundamental [6]. Beutler [4] has shown that the Poisson sampling scheme is alias free relative to the family of all spectral distributions. Then, in principle, the process can be reconstructed from Poisson samples; so in theory such a high-order interpolation should exist.

The problem of optimum interpolation consists of finding an operation $\mathcal{F}_{t,\{t_n\}}$ such that

$$y(t) = \mathcal{F}_{t,\{t_n\}}\Big[\big\{x(t_n)\big\}\Big] \qquad (2)$$

is the "best" recovery of $x(t)$, for all t.
The zero-order hold is defined by

$$y(t) = x(t_n), \quad t_n \leq t < t_{n+1}. \qquad (3)$$

This is the classical stepwise reconstruction procedure.
The polygonal hold reconstructor defines the approximation function $y(t)$ as

$$y(t) = \frac{(t-t_n)x(t_{n+1}) + (t_{n+1}-t)x(t_n)}{t_{n+1}-t_n}, \; t_n < t \leq t_{n+1}. \qquad (4)$$

2.2 Mean Square Comparison

In the case of the zero-order hold, the mean-square error is

$$\begin{aligned} \overline{\varepsilon^2} &= E\Big[\big\{y_0(t) - x(t)\big\}^2\Big] \\ \overline{\varepsilon^2} &= 2R_x(0) - 2\lambda \int_0^\infty R_x(\sigma) e^{-\lambda\sigma} d\sigma, \end{aligned} \qquad (5)$$

where σ denotes the length of time elapsed between the last sampling time before t and t itself, and $R_x(s)$ is the correlation function of the original signal $x(t)$.
In the case of the polygonal hold, the mean square error is

$$\begin{aligned} \overline{\varepsilon^2} &= R_x(0)\Big[1 + 2\lambda^2 \int_0^\infty \int_0^\infty \frac{\sigma^2}{(\sigma+\beta)^2} e^{-\lambda(\sigma+\beta)} d\sigma d\beta\Big] \\ &- 4\lambda^2 \int_0^\infty \int_0^\infty \frac{\beta}{\sigma+\beta} e^{-\lambda(\sigma+\beta)} R_x(\sigma) d\sigma d\beta. \qquad (6) \\ &+ 2\lambda^2 \int_0^\infty \int_0^\infty \frac{\sigma\beta}{(\sigma+\beta)^2} e^{-\lambda(\sigma+\beta)} R_x(\sigma+\beta) d\sigma d\beta \end{aligned}$$

Performances of Poisson sampling are elsewhere studied for various interpolators when the input signal is a band-limited white noise [3], [7], [8]. Comparing the simplest interpolator, we can conclude that the first-order hold is much better than the zero-order hold.
Moreover, for high sampling frequencies the polygonal hold is almost as good as the linear (nonrealizable) interpolator and performs better than the exponential (see Host-Madsen et al. (5) for the general exponential interpolation).

3 SPECTRAL ANALYSIS OF RANDOM SAMPLES

3.1 General formulation

It is well known that the spectrum of a periodic impulse train is another periodic impulse train in the frequency domain. However, the spectrum of a set of random (or nonuniform) impulses, in general, is not another set of impulses. This implies that the spectrum of random samples might be totally smeared (aliased) in the frequency domain and we cannot use any linear time-invariant filter for signal recovery.

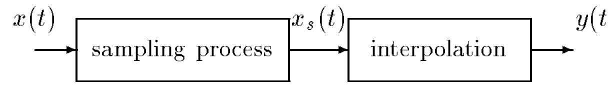

Figure 1: Sampling process and interpolation

If $p(t) = \sum_i \delta(t-t_i)$ is any random sequence of impulses and $x(t)$ is the original signal, then the sampled signal is

$$x_s(t) = p(t) \cdot x(t).$$

Therefore, the power spectrum of the randomly sampled signal $x_s(t)$ is

$$S_s(f) = S_p(f) * S_x(f), \qquad (7)$$

where $*$ is the convolution operator.
In the case of a Poisson impulsive sampling, the power spectrum of the point process $p(t)$, is

$$S_p(f) = \lambda + \lambda^2\delta(f). \qquad (8)$$

Therefore, substituting equation (8) into (7), the power spectrum of the sampled signal is

$$S_s(f) = \lambda^2 S_x(f) + N, \qquad (9)$$

where N equals $\lambda R_x(0)$.
The power spectrum of random samples (Poisson impulsive sampling) of a random signal resembles that of the signal imbedded in a "white" noise. Thus in the frequency domain, we generally expect the original signal to be corrupted by noise. We conclude that the simple division by λ (after sampling) yields a good estimate of the signal.
This leads to the new expression of the power spectral density of the random sampled signal:

$$S_s(f) = S_x(f) + \frac{R_x(0)}{\lambda}. \qquad (10)$$

Equation (10) confirms the intuitive result that by increasing the density of sampling rate, the SNR (signal to noise ration) improves and the PSD of the sampled signal is unbiased as $\lambda \longrightarrow \infty$.

3.2 Interpolation

From equation (2), we can conclude that the impulse response of the interpolator is a time varying function. The reason is that if the input is delayed, the output is not equal to its delayed version.
Note that if the sampling process is uniform, the impulse response of the interpolator is a *sinc* function representing an ideal low-pass filter, the output is equal to the input and the system is a linear time varying system. In practice, there are some reconstruction methods. We mention only the most promising ones, namely, the zero-order interpolator (sample-and-hold), the exponential hold and the polygonal hold.
Intuitively, we state that the output of any practical interpolator can be expressed by

$$y(t) = h(t) * \big(x(t) + b(t)\big), \qquad (11)$$

where $h(t)$ is an impulse response representing a low-pass filter, $x(t)$ is the original (continuous) signal and $b(t)$ is a noise resulting from both the random sampling process and the interpolation ($x(t)$ and $b(t)$ are then supposed independent). From which we conclude that the power spectral density of the interpolated signal is

$$S_y(f) = \mid H(f) \mid^2 \left(S_x(f) + S_b(f)\right), \qquad (12)$$

where $H(f)$ is the transfer function of the low-pass filter related to $h(t)$ by Fourier transform and $S_b(f)$ is the PSD of $b(t)$.

Boyer et al, [2] and Adrian et al, [1] calculated the spectrum of a signal derived from a continuous signal by "sample-and-hold" random sampling and confirmed our general formulation given in (12). They have shown that the effect of the sample-and-hold process is to low pass filter the true spectrum with low pass frequency λ and the additive term represents the spectrum of a white noise filtered by the same low-pass filter. The low-pass filtering is caused by the information loss that occurs during the hold instants, and the white noise is created by the random steps that occur at new samples.

Recently, Host-Madsen et al [5] have shown that the effect of exponential interpolation can be expressed as formulation (12):" the effect of exponential interpolation is that a constant is added to the spectrum, and the sum is multiplied by a first-order low-pass filter with cut-off frequency $\lambda + b$", where b is the parameter in the exponential interpolator. Note that for $b = 0$, we get the well-known sample and hold interpolation.

Based on these resuts, we shall experimentally show in the next section that, in the case of the polygonal interpolator, the filter is also a low-pass one but the additive term does not represente a white noise.

4 EXPERIMENTAL RESULTS

The first test signal used is a band-limited noise. This signal is randomly sampled at the average rate $\lambda = 10\ kHz$. The estimated spectra are given after averaging on 50 realizations using the two interpolation schemes (zero-order and first-order holds). These results are compared to the theoretical spectrum via the transfer function of the low-pass filter (Fig.2). We note that, at low frequencies, the filter does not depend on the interpolation method. Then, the noise resulting from the interpolation is estimated (Fig. 3 and 4). Finally, the compensated spectra are compared to the original one (Fig.5). It can be seen, on this last figure, that the zero-order hold gives a uniform bias, whereas the polygonal hold has a better behavior at high frequencies; at low frequencies, the two interpolation schemes show similar behaviours. Fig.4 confirms that, in the sample-and-hold case, the estimation introduces a white noise, but Fig.4 shows that, in the polygonal hold case, this noise is low-pass filtered. We can see in Fig.6 that the PSD of a Kolmogorov signal is better estimated in high frequencies using the polygonal interpolator. These results are also confirmed with other signals such as narrow-band noise, $1/f$ noise,...

5 CONCLUSION

There are a number of open theoretical questions, for example regarding the nature of aliasing in estimates from an arbitrary sampling scheme, the exact analytic expression of the power spectrum in the case of a first-order interpolator. There are a number of different methods available, but little practical experience to show their relative merits and assist in choosing which should be applied to a particular problem. In this paper, we have investigated the power spectrum estimation from randomly sampled signals and obtained preliminar results assuming that the sampling times are Poisson distributed, with a constant average sampling rate. Indeed, in the majority of practical situations, this distribution will be a good model. We have shown that the effect of the additive noise is significantly reduced by a firt-order interpolator, especially at high frequencies where the distorsion is lower than for zero-order "sample-and-hold" processor.

Although, this study was motivated primarily by the need to analyze the experimental information mentioned above [9], the ideas upon which it is based are of a fundamental nature and as such are likely to have a wide area of applications: communications, control, biomedical, geophysics, ...

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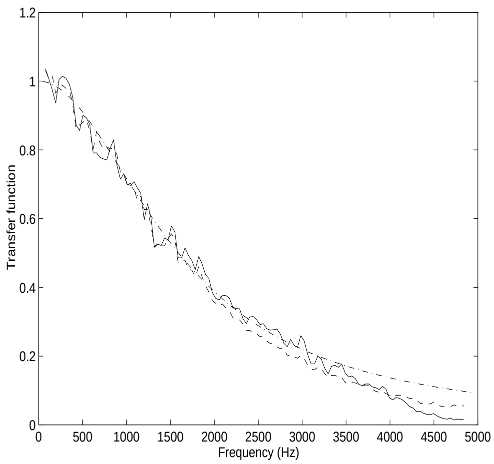

Figure 2: *Transfer function resulting from the interpolation of a randomly sampled band-limited noise. (- -): zero-order, (-.-): modelling, (–): first-order.*

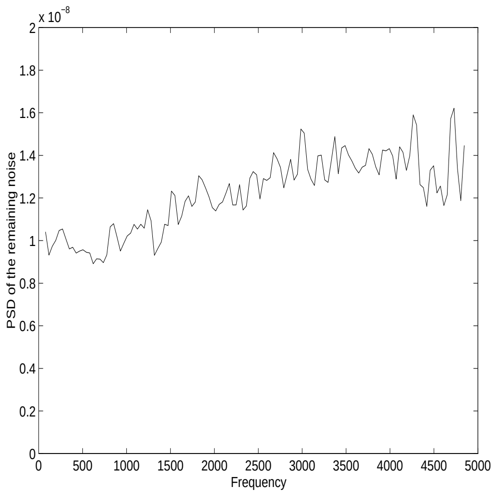

Figure 3: *Estimation of the PSD of the noise resulting from the interpolation of a randomly sampled band-limited noise using a sample-and-hold.*

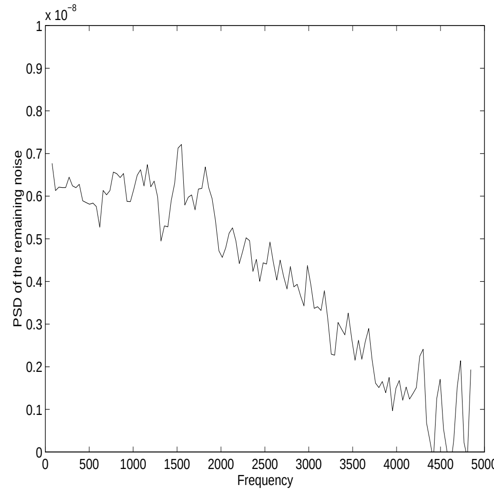

Figure 4: *Estimation of the PSD of the noise resulting from the interpolation of a randomly sampled band-limited noise using a polygonal hold.*

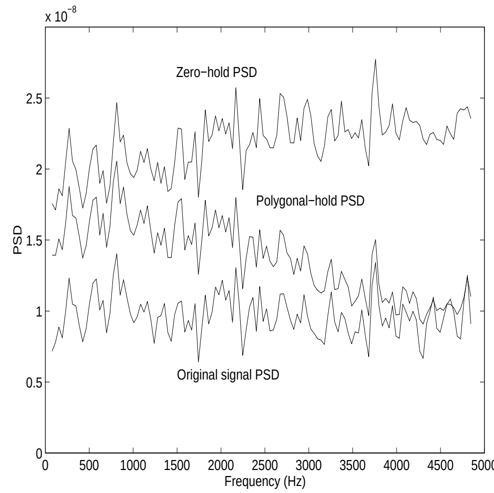

Figure 5: *Estimation of the PSD of the signal test (band-limited noise).*

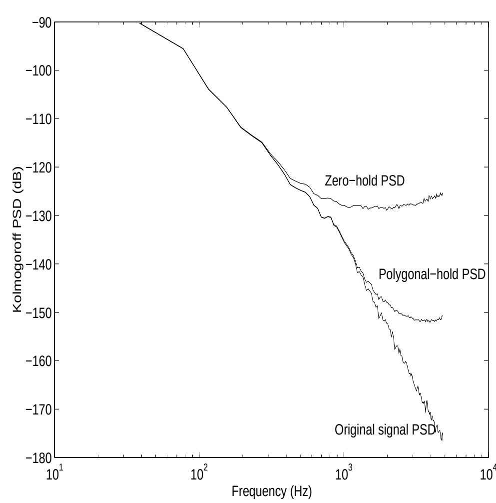

Figure 6: *Estimation of the PSD of a Kolmogorov process.*