SUPER-RESOLUTION SPECTRUM ANALYSIS REGULARIZATION : BURG, CAPON & AGO-ANTAGONISTIC ALGORITHMS

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ABSTRACT

We propose a regularized Burg algorithm, based on a frequency domain smoothness prior constraint, which solves model order estimation problem in case of short data records. A second algorithm deals with a recursive eigendecomposition method from autoregressive parameters, that allows Capon spectrum analysis regularization. Finally, we have developed a new regularized detectors using log-likehood ratio from regularized reflection coefficients.

1. PREAMBLE

We have developed a regularized Burg algorithm [1] based on Bayesian spectrum analysis with a frequency domain smoothness prior constraint. This spectrum analysis is well adapted when only short data records are available, could be formulated as an ill-posed problem and solved by a Tikhonov approach. This regularized Burg algorithm preserves lattice structure but with a new regularized reflection coefficient calculated from autoregressive parameters. This algorithm is compared with Burg-MFE algorithm [2].

We have extended these results to super-resolution algorithms using eigenvector-based projections. First, we have identified the inverse autocorrelation matrix and expressed it from autoregressive parameters and , secondly, established a recursive relation between eigenvalues, eigenvectors and autoregressive parameters. Contrary to RISE algorithm [3], based on correlation parameters, our explicit solution from autoregressive parameters allows an eigendecomposition regularization.

2. MAXIMUM ENTROPY REGULARIZATION

Autoregressive parameters are provided by Yule-Walker equation :

$$\Omega_n . A_n = -C_n \quad \text{with} \quad \Omega_n = \begin{bmatrix} c_0 & c_1^* & \cdots & c_{n-1}^* \\ c_1 & c_0 & \cdots & c_{n-2}^* \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix}$$

$$A_n = \left[a_1^{(n)} \; \ldots \; a_n^{(n)} \right]^T \quad \text{and} \quad c_k = E\left[x_m . x_{m-k}^* \right]$$

This normal equation has been solved by Levinson with a recursive-in-order algorithm :

$$a_k^{(n)} = a_k^{(n-1)} + a_n^{(n)} . a_{n-k}^{(n-1)*} \qquad k = 1, \ldots, n-1$$

Finally, Burg has proposed a simpler algorithm that induces a lattice structure filter. We examine a new version algorithm that allows to free Burg algorithm from model order estimation.

2.1 Classical Burg Algorithm

At each order recursion, the arithmetic mean of the forward and backward linear prediction error power is minimized, subject to the recursion given by Levinson Equation on AR parameters. Prediction errors are defined by :

$$\begin{cases} f_m(n) = \sum_{k=0}^{m} a_k^{(m)} . x_{n-k} \\ b_m(n) = \sum_{k=0}^{m} a_k^{(m)*} . x_{n-m+k} \end{cases} \quad \text{with} \quad a_0^{(m)} = 1$$

$$A_n = \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} + \mu_n . \begin{bmatrix} A_{n-1}^{(-)} \\ 1 \end{bmatrix} \quad \text{with} \quad \mu_n = a_n^{(n)}$$

$$\text{and} \quad A_n^{(-)} = \left[a_n^{(n)*} \; \ldots \; a_1^{(n)*} \right]^T$$

$$P^{(m)} = \sum_{n=m+1}^{N} \left| f_m(n) \right|^2 + \left| b_m(n) \right|^2 = \left[1 - \left| \mu_m \right|^2 \right] . P^{(m-1)}$$

This minimization provides the estimate of the reflection coefficient which is the harmonic mean between the forward and backward partial correlation coefficients :

$$E^{(m)} = U^{(m)} \quad \text{with} \quad U^{(m)} = \frac{1}{2.(N-m)} \sum_{n=m+1}^{N} \left| f_m(n) \right|^2 + \left| b_m(n) \right|^2$$

$$\nabla_{\mu_m} U^{(m)} = \mu_m . G^{(m)} + D^{(m)*} = 0 \Rightarrow \mu_m = -\frac{D^{(m)*}}{G^{(m)}}$$

$$\text{with} \begin{cases} G^{(m)} = \frac{1}{N-m} \sum_{n=m+1}^{N} \left| f_{m-1}(n) \right|^2 + \left| b_{m-1}(n-1) \right|^2 \\ D^{(m)} = \frac{2}{N-m} \sum_{n=m+1}^{N} b_{m-1}(n-1) . f_{m-1}^*(n) \end{cases}$$

2.2 Burg Minimum Free Energy Algorithm

Silverstein has developed a modified Burg algorithm based on an analogy with statistical thermodynamic. The minimum free energy algorithm provides the relection coefficient by minimizing the free energy at each stage in the Levinson recursion, with the energy always defined as Burg, and the entropy represented by the Shannon Burg form :

$$E^{(m)}=U^{(m)}-\alpha.H^{(m)} \quad \text{with} \quad H^{(m)}=\int_{-1/2}^{1/2}\ln\left[\frac{P^{(m)}}{\left|A^{(m)}(f)\right|^2}\right].df$$

$$A^{(m)}(f)=\sum_{k=0}^{m}a_k^{(m)}e^{-j\omega k}=A^{(m-1)}(f)+\mu_m e^{-j\omega m}A^{(m-1)*}(f)$$

$$\nabla_{\mu_m}H^{(m)}=\frac{\nabla_{\mu_m}P^{(m)}}{P^{(m)}}-2\,\mathrm{Re}\left[\int_{-1/2}^{1/2}\frac{\nabla_{\mu_m}A^{(m)*}(f)}{A^{(m)*}(f)}.df\right]$$

with $\nabla_{\mu_m}A^{(m)*}(f)=2.e^{j2\pi fm}.A^{(m-1)}(f)$

zero equality of last integral implies assumption that reflection coefficient modulus are all lower than unity.Then:

$$\nabla_{\mu_m}H^{(m)}=\frac{\nabla_{\mu_m}P^{(m)}}{P^{(m)}}=\frac{-2.\mu_m}{1-\left|\mu_m\right|^2}$$

because $P^{(m)}=\left[1-\left|\mu_m\right|^2\right]P^{(m-1)}$ and $\nabla_{\mu_m}\left|\mu_m\right|^2=2.\mu_m$

Minimum free energy algorithm leads to solve zeros of a 3rd degree polynomial with real coefficients :

$$\nabla_{\mu_m}E^{(m)}=0\Rightarrow D^{(m)*}+\mu_m.G^{(m)}=\frac{-2.\alpha.\mu_m}{1-\left|\mu_m\right|^2}$$

Acceptable solution corresponds to the zero associated with reflection coefficient modulus lower than unity, to make sure of first assumption validity :

$$\mu_m=\frac{\xi_m.D^{(m)*}}{\left|D^{(m)}\right|} \quad \text{with } \xi_m \text{ zero of :}$$

$$\left(1\text{-}\xi^2\right).\left(\xi_m.G^{(m)}+\left|D^{(m)}\right|\right)=-2.\alpha.\xi_m \quad \text{with} \quad \left|\mu_m\right|<1$$

2.2 Regularized Burg Algorithm

The regularised Burg algorithm uses Levinson recursive-in-order equation, which allows to develop cost function gradient, and to express the new reflection coefficient according to additional spectral smoothness constraints

$$E^{(m)}=U^{(m)}+\sum_{k=0}^{1}\gamma_k M_k^{(m)} \text{with } M_k^{(m)}=\int_{-1/2}^{1/2}\left|\frac{d^kA^{(m)}(f)}{df^k}\right|^2df$$

$$\nabla_{\mu_m}E^{(m)}=\nabla_{\mu_m}U^{(m)}+\gamma_0.\nabla_{\mu_m}M_0^{(m)}+\gamma_1.\nabla_{\mu_m}M_1^{(m)}$$

$$\textbf{with}\begin{cases}\mathbf{D_{reg}^{(m)}}=\mathbf{D^{(m)}}+\left[\mathbf{2.}\sum_{\mathbf{k=1}}^{\mathbf{m-1}}\beta_{\mathbf{k}}^{\mathbf{(m)}}\mathbf{.a_k^{(m-1)}.a_{m-k}^{(m-1)}}\right]^* \\ \mathbf{G_{reg}^{(m)}}=\mathbf{G^{(m)}}+\mathbf{2.}\sum_{\mathbf{k=0}}^{\mathbf{m-1}}\beta_{\mathbf{k}}^{\mathbf{(m)}}.\left|\mathbf{a_k^{(m-1)}}\right|^2\end{cases}$$

$$\mathbf{m_m}=-\frac{\mathbf{D_{reg}^{(m)*}}}{\mathbf{G_{reg}^{(m)}}} \quad \textbf{and} \quad \beta_{\mathbf{k}}^{\mathbf{(m)}}=\gamma_{\mathbf{0}}+\gamma_{\mathbf{1}}\mathbf{.(2.}\pi\mathbf{)^2.(k-m)^2}$$

Gradient cancellation provides final reflection coefficient of regularized Burg algorithm. This algorithm could be interpreted and linked with regularized mean square algorithm by a statistical interpretation of Burg regularization :

$$\begin{cases}D_{reg}^{(m)*}=2.E\left[f_{m-1}(n).b_{m-1}^*(n-1)\right]+2.\sum_{k=1}^{m-1}\beta_k^{(m)}.a_k^{(m-1)}.a_{m-k}^{(m-1)} \\ G_{reg}^{(m)}=E\left[\left|f_{m-1}(n)\right|^2\right]+E\left[\left|b_{m-1}(n-1)\right|^2\right]+2.\sum_{k=1}^{m-1}\beta_k^{(m)}.\left|a_k^{(m-1)}\right|^2\end{cases}$$

$$\mu_m=-\frac{\sum_{l=0}^{m-1}\sum_{k=1}^{m}a_l^{(m-1)}.a_{m-k}^{(m-1)}.\bar{c}_{k-l}}{\sum_{l=0}^{m-1}\sum_{k=0}^{m-1}a_l^{(m-1)}.a_k^{(m-1)*}.\bar{c}_{k-l}} \; , \; \begin{cases}\bar{c}_{k-l}=c_{k-l} \text{ if } k\neq l \\ \bar{c}_{k-l}=c_{k-l}+\beta_k^{(m)} \text{ oth.}\end{cases}$$

Hence, the new implicit autocorrelation matrix is provided ,as for the mean square approach, by :

$$\Omega_n^{reg}=\Omega_n+\gamma_0.I_n+\gamma_1.(2.\pi)^2.\Theta_n$$

$$\text{with } \Theta_n=\begin{bmatrix}n^2 & 0 & ... & 0 \\ 0 & (n-1)^2 & 0\,... & 0 \\ ... & ... & ... & ... \\ 0 & ... & 0 & 1^2\end{bmatrix}$$

2.3 Ago-antagonistic Burg Algorithm

To increase resolution of regularized Burg algorithm and avoid smoothness effect of regularization, we have added an additional constraint that attracts poles to unit circle (agonistic with data and antagonistic with regularization).

$$E^{(m)}=U^{(m)}+\sum_{k=0}^{1}\gamma_k.M_k^{(m)}+\delta.Ln\left[1-\left|\mu_m\right|^2\right]$$

$$\text{with } \mu_m=(-1)^m.\prod_{i=1}^{m}z_i^{(m)} \text{ and } \nabla_{\mu_m}Ln\left[1-\left|\mu_m\right|^2\right]=\frac{-2.\mu_m}{1-\left|\mu_m\right|^2}$$

As Burg-MFE, this algorithm is solved by the search for zeros of 3rd order real polynomial :

$$D_{reg}^{(m)*}+\mu_m.G_{reg}^{(m)}=\frac{2.\delta.\mu_m}{1-\left|\mu_m\right|^2} \quad \text{but} \quad \mu_m.D_{reg}^{(m)}\in\Re$$

$$\text{we set } \xi_m=\frac{\mu_m.D_{reg}^{(m)}}{\left|D_{reg}^{(m)}\right|} \; , \; \left|\xi_\mathrm{m}\right|<1 \text{ zero of}$$

$$\left(1\text{-}\xi_\mathrm{m}^2\right).\left(\xi_m.G_{reg}^{(m)}+\left|D_{reg}^{(m)}\right|\right)=2.\delta.\xi_m \quad \text{and} \quad \mu_m=\frac{\xi_m.D_{reg}^{(m)*}}{\left|D_{reg}^{(m)}\right|}$$

δ is optimal when the straight line of the right term is tangential to 3rd order polynomial of the left term :

$$\begin{cases}Q(\xi_m)=\left(1-\xi_m^2\right).\left(\xi_m.G_{reg}^{(m)}+\left|D_{reg}^{(m)}\right|\right)=2.\delta_{opt}.\xi_m \\ \dfrac{dQ(\xi_m)}{d\xi_m}=2.\delta_{opt}\end{cases}$$

Reflection coefficient solution of ago-antagonistic algorithm could be obtained recursively by mean of substitution method, because :

$$\frac{\left|D_{reg}^{(m)}\right|}{G_{reg}^{(m)}}<1\Rightarrow\left|\frac{dT(\xi_m)}{d\xi_m}\right|<1$$

$$\begin{cases} \xi_{\mathbf{m,k+1}} = \left[- \dfrac{\left|\mathbf{D_{reg}^{(m)}}\right| . \left(\xi_{\mathbf{m,k}}^{2} + \mathbf{1}\right)}{\mathbf{2.G_{reg}^{(m)}}} \right]^{\mathbf{1/3}} = \mathbf{T}\left(\xi_{\mathbf{m,k}}\right), \xi_{\mathbf{m,0}} = - \dfrac{\left|\mathbf{D_{reg}^{(m)}}\right|}{\mathbf{G_{reg}^{(m)}}} \\ \mathbf{d_{opt}} = \dfrac{\mathbf{G_{reg}^{(m)}} . \left(\mathbf{1} - \xi_{\mathbf{m}}^{2}\right)^{2}}{\mathbf{2.}\left(\mathbf{1} + \xi_{\mathbf{m}}^{2}\right)} < \dfrac{\mathbf{G_{reg}^{(m)}}}{\mathbf{2}} \ \textbf{because} \ \left|\xi_{\mathbf{m}}\right| < \mathbf{1} \end{cases}$$

We prove that this optimal solution maintains positivity of the new cost function :

$$Ln(x) > (x-1) \Rightarrow \left[E^{(m)} > 0 \Leftrightarrow \delta < \frac{U^{(m)} + \sum_{k=0}^{1} \gamma_k . M_k^{(m)}}{\left|\mu_m\right|^2} \right]$$

$$\text{but} \quad U^{(m)} + \sum_{k=0}^{1} \gamma_k . M_k^{(m)} > \frac{G_{reg}^{(m)}}{2} \ \text{and} \ \left|\mu_m\right| < 1$$

$$\delta_{opt} < \frac{G_{reg}^{(m)}}{2} < \frac{U^{(m)} + \sum_{k=0}^{1} \gamma_k . M_k^{(m)}}{\left|\mu_m\right|^2} \Rightarrow E^{(m)} > 0$$

3. EIGENVECTOR-BASED REGULARIZATION

3.1 Capon Algorithm

An other spectrum analysis approach is based on eigen space decomposition, and provides Capon spectrum :

$$S_{Capon}^{(n)}(f) = \left[e^{(n)+} . \Omega_n^{-1} . e^{(n)} \right]^{-1} = \frac{1}{\sum_{k=1}^{n} \frac{1}{\lambda_k^{(n)}} . \left| X_k^{(n)+} . e^{(n)} \right|^2}$$

$$\text{with } e^{(n)} = \begin{bmatrix} 1 & e^{-j\omega} & .... & e^{-j(n-1)\omega} \end{bmatrix}^T$$

In case of short data records, this problem is also ill-posed and we propose a regularization algorithm from previous autoregressive parameters.

3.2 Weighted Capon Algorithm

Caspary & al have proposed a weighted Capon algorithm based on a combination of silverstein and Choi's weighting, which reaches a compromise between separate signal and noise subspace criteria, and a matrix conditioning criteria :

$$\rho_i = \frac{1}{\lambda_i} \ \text{ replaced by } \ \rho_i = \left[\left[\frac{\lambda_i^{(n)}}{\lambda_n^{(n)}} \right]^p + \left[\frac{\bar{\lambda}^{(n)}}{\lambda_n^{(n)}} \right] \right]^{-1}$$

3.3 Regularized Capon Algorithm

According to our previous AR regularization, regularized eigenvalues and eigenvectors are not directly deduced, because the new regularized autocorrelation matrix could not be simply diagonalized

$$U_n^+ . \Omega_n^{reg} . U_n = \Gamma_n + \gamma_0 . I_n + \gamma_1 . (2\pi)^2 . U_n^+ . \Theta_n . U_n$$

$$U_n = \left[X_1^{(n)} ... X_n^{(n)} \right] \text{ with } U_n^+ . U_n = U_n . U_n^+ = I_n$$

$$\text{and } \Gamma_n = diag\left\{ \lambda_1^{(n)}, ..., \lambda_n^{(n)} \right\}$$

So, we propose to express eigen parameters, recursively, from regularized AR parameters, thanks to a Trench algorithm extension.

3.3.1 Trench Algorithm Extension

Trench has identified structure of the inverse autocorrelation matrix :

$$\Phi_n = \Omega_n^{-1} = \begin{bmatrix} \alpha_{n-1} & E_{n-1}^+ \\ E_{n-1} & \Phi_{n-1} + \frac{1}{\alpha_{n-1}} . E_{n-1} . E_{n-1}^+ \end{bmatrix}$$

So, we have expressed a more precise formulation of this inverse structure from AR parameters and the linear prediction error power :

$$\begin{cases} \begin{bmatrix} \alpha_{n-1} & E_{n-1}^+ \\ E_{n-1} & \Phi_{n-1} + \frac{1}{\alpha_{n-1}} . E_{n-1} . E_{n-1}^+ \end{bmatrix} . \begin{bmatrix} c_0 & C_{n-1}^+ \\ C_{n-1} & \Omega_{n-1} \end{bmatrix} = I_n \\ \Omega_{n-1}^{-1} . C_{n-1} = -A_{n-1} \end{cases}$$

$$\begin{cases} \mathbf{\Phi_n} = \mathbf{\Omega_n^{-1}} = \begin{bmatrix} \boldsymbol{\alpha_{n-1}} & \boldsymbol{\alpha_{n-1}} . \mathbf{A_{n-1}^+} \\ \boldsymbol{\alpha_{n-1}} . \mathbf{A_{n-1}} & \mathbf{\Phi_{n-1}} + \boldsymbol{\alpha_{n-1}} . \mathbf{A_{n-1}} . \mathbf{A_{n-1}^+} \end{bmatrix} \\ \boldsymbol{\alpha_{n-1}} = \left[\mathbf{c_0} + \mathbf{C_{n-1}^+} . \mathbf{A_{n-1}} \right]^{-1} \end{cases}$$

α corresponds to the inverse linear prediction error power :

$$\begin{cases} a_k^{(n)} = a_k^{(n-1)} + \mu_n . a_{n-k}^{(n-1)*} \\ a_n^{(n)} = - \dfrac{\left[c_n^* + C_{n-1}^+ . A_{n-1}^{(-)} \right]^*}{c_0 + C_{n-1}^+ . A_{n-1}} \Rightarrow \alpha_n^{-1} = \left[1 - \left| a_n^{(n)} \right|^2 \right] . \alpha_{n-1}^{-1} \end{cases}$$

$$\boldsymbol{\alpha_n^{-1}} = \left[\mathbf{1} - \left| \boldsymbol{\mu_n} \right|^{\mathbf{2}} \right] . \boldsymbol{\alpha_{n-1}^{-1}} = \mathbf{P^{(n)}} \quad \text{with}$$

$$P^{(n)} = \frac{1}{2} . E\left[\left| f_n(m) \right|^2 + \left| b_n(m) \right|^2 \right] = \alpha_n^{-1} \ \text{ and } \ c_0 = \alpha_0^{-1} = P^{(0)}$$

We recognize the triangular decomposition of Choleski :

$$\Phi_n = \sum_{k=0}^{n-1} \alpha_k . V_k^{(n)} . V_k^{(n)+} = W_n^+ . diag\{ ..., \alpha_{n-k-1}, ... \} . W_n$$

$$W_n = \left[V_{n-1}^{(n)} ... V_0^{(n)} \right]^+ \ \text{ and } \ V_k^{(n)} = \begin{bmatrix} 0_{n-k-1} & 1 & A_k \end{bmatrix}^T$$

3.3.2 Recursive/Iterative Eigen-decomposition Regularization via autoregressive parameters

We diagonalize the inverse autocorrelation matrix from its expression with autoregressive parameters and inverse linear prediction error power :

$$\Phi_n . U_n = U_n . \Lambda_n \ \text{ with } \ \Lambda_n = diag\left\{ \eta_1^{(n)}, ..., \eta_n^{(n)} \right\}, \ \lambda_i^{(n)} = \frac{1}{\eta_{n-i+1}^{(n)}}$$

$$\begin{bmatrix} \alpha_{n-1} - \eta_i^{(n)} & \alpha_{n-1} . A_{n-1}^+ \\ \alpha_{n-1} . A_{n-1} & \Phi_{n-1} - \eta_i^{(n)} . I_{n-1} + \alpha_{n-1} . A_{n-1} . A_{n-1}^+ \end{bmatrix} \begin{bmatrix} 1 \\ X_{[n-1],i}^{(n)} \end{bmatrix} = 0_n$$

$$\text{with } \ X_i^{(n)} = \begin{bmatrix} 1 \\ X_{[n-1],i}^{(n)} \end{bmatrix}$$

$$\Rightarrow \begin{cases} \eta_i^{(n)} = \alpha_{n-1} . \left[1 + A_{n-1}^+ . X_{[n-1],i}^{(n)} \right] & (1) \\ X_{[n-1],i}^{(n)} = -\eta_i^{(n)} . \left[\Phi_{n-1} - \eta_i^{(n)} . I_{n-1} \right]^{-1} . A_{n-1} & (2) \end{cases}$$

$$(1) \Rightarrow \eta_i^{(n)}.\left[1+\alpha_{n-1}.A_{n-1}^{+}.\left[\Phi_{n-1}-\eta_i^{(n)}.I_{n-1}\right]^{-1}.A_{n-1}\right]=\alpha_{n-1}$$

but $\Phi_{n-1}=U_{n-1}.\Lambda_{n-1}.U_{n-1}^{+}$ with $U_{n-1}.U_{n-1}^{+}=I_{n-1}$

$$\frac{\eta_i^{(n)}}{\alpha_{n-1}}\left[1+\alpha_{n-1}\left(U_{n-1}^{+}A_{n-1}\right)^{+}\left[\Lambda_{n-1}-\eta_i^{(n)}I_{n-1}\right]^{-1}U_{n-1}^{+}A_{n-1}\right]=1$$

So, inverse eigenvalues are zeros of a function that depends on inverse eigenvalues and eigenvectors at direct inferior order and autoregressive parameters :

$$\mathbf{F^{(n)}}\left(\eta_{\mathbf{i}}^{(\mathbf{n})}\right)=\eta_{\mathbf{i}}^{(\mathbf{n})}-\alpha_{\mathbf{n-1}}+\alpha_{\mathbf{n-1}}.\eta_{\mathbf{i}}^{(\mathbf{n})}.\sum_{\mathbf{k=1}}^{\mathbf{n-1}}\frac{\left|\mathbf{A}_{\mathbf{n-1}}^{+}.\mathbf{X}_{\mathbf{k}}^{(\mathbf{n-1})}\right|^{2}}{\left(\eta_{\mathbf{k}}^{(\mathbf{n-1})}-\eta_{\mathbf{i}}^{(\mathbf{n})}\right)}=\mathbf{0}$$

Derivative of this function is strictly greater than unity :

$$\frac{\partial F^{(n)}(\eta)}{\partial\eta}=1+\alpha_{n-1}.\sum_{k=1}^{n-1}\frac{\eta_k^{(n-1)}.\left|A_{n-1}^{+}.X_k^{(n-1)}\right|^{2}}{\left(\eta_k^{(n-1)}-\eta\right)^{2}}>1$$

$$\frac{\partial\mathbf{F^{(n)}}(\eta)}{\partial\eta}>\mathbf{1}\quad,\quad\forall\eta\in\bigcup_{\mathbf{k=1}}^{\mathbf{n-1}}\left\{\right]\eta_{\mathbf{k+1}}^{(\mathbf{n-1})},\eta_{\mathbf{k}}^{(\mathbf{n-1})}\left[\right\}$$

If we apply corollaire of Courant-Fisher theorem, it proves the interlacing of eigenvalues at different orders, because autocorrelation matrix of order (n-1) is included in matrix of order n :

$$\eta_n^{(n)}<\eta_{n-1}^{(n-1)}<\eta_{n-1}^{(n)}<\ldots<\eta_2^{(n)}<\eta_1^{(n-1)}<\eta_1^{(n)}$$

upper limit is provided by trace of the inverse autocorrelation matrix :

$$0<\eta_n^{(n)}<\ldots<\eta_{\mathbf{1}}^{(\mathbf{n})}<\mathbf{Trace}\left[\Phi_{\mathbf{n-1}}\right]\text{ but with } T_{n-1}=\begin{bmatrix}1 & A_{n-1}\end{bmatrix}^{T}$$

$$\mathbf{Trace}\left[\Phi_{\mathbf{n}}\right]=\mathbf{Trace}\left[\Phi_{\mathbf{n-1}}\right]+\alpha_{\mathbf{n-1}}.\mathbf{T}_{\mathbf{n-1}}^{+}.\mathbf{T}_{\mathbf{n-1}}$$

If we exploit the second equation from inverse autocorrelation matrix diagonalization :

$$(2)\Rightarrow X_{[n-1],i}^{(n)}=-\eta_i^{(n)}.\left[\Phi_{n-1}-\eta_i^{(n)}.I_{n-1}\right]^{-1}.A_{n-1}$$

$$\mathbf{X}_{[\mathbf{n-1}],\mathbf{i}}^{(\mathbf{n})}=-\eta_{\mathbf{i}}^{(\mathbf{n})}.\mathbf{U}_{\mathbf{n-1}}.\mathbf{diag}\left\{\ldots,\frac{\mathbf{1}}{\eta_{\mathbf{k}}^{(\mathbf{n-1})}-\eta_{\mathbf{i}}^{(\mathbf{n})}},\ldots\right\}.\mathbf{U}_{\mathbf{n-1}}^{+}.\mathbf{A}_{\mathbf{n-1}}$$

$$\textbf{with}\quad\mathbf{U}_{\mathbf{n-1}}=\left[\mathbf{X}_{\mathbf{1}}^{(\mathbf{n-1})}\ldots\mathbf{X}_{\mathbf{n-1}}^{(\mathbf{n-1})}\right]\quad\textbf{and}\quad\mathbf{X}_{\mathbf{i}}^{(\mathbf{n})}=\begin{bmatrix}\mathbf{1} & \mathbf{X}_{[\mathbf{n-1}],\mathbf{i}}^{(\mathbf{n})}\end{bmatrix}^{\mathbf{T}}$$

After each recursion, eigenvectors have to be normalized :

$$\mathbf{X}_{\mathbf{i}}^{(\mathbf{n})}=\frac{\mathbf{X}_{\mathbf{i}}^{(\mathbf{n})}}{\left|\mathbf{X}_{\mathbf{i}}^{(\mathbf{n})}\right|}\Rightarrow U_n^{+}.U_n=U_n.U_n^{+}=I_n$$

3.3.3 Additional Result

An additional result deals with a relation between inverse eigenvalue and derivative of F(.) :

$$\eta_i^{(n)}=\frac{X_i^{(n)+}.\Phi_n.X_i^{(n)}}{X_i^{(n)+}.X_i^{(n)}}\quad\text{with}\quad X_i^{(n)+}.X_k^{(n)}=\delta_{i,k}$$

$$\eta_i^{(n)}=\frac{\eta_i^{(n)2}.\left|X_{i,1}^{(n)}\right|^{2}}{\alpha_{n-1}}\left[1+\alpha_{n-1}.H_{n-1}^{+}.\Lambda_{n-1}.H_{n-1}\right]$$

with $H_{n-1}=\left[\Lambda_{n-1}-\eta_i^{(n-1)}.I_{n-1}\right]^{-1}.U_{n-1}^{+}.A_{n-1}$

$$\frac{\partial\mathbf{F^{(n)}}\left(\eta_{\mathbf{i}}^{(\mathbf{n})}\right)}{\partial\eta}=\frac{\alpha_{\mathbf{n-1}}}{\eta_{\mathbf{i}}^{(\mathbf{n})}.\left|\mathbf{X}_{\mathbf{i,1}}^{(\mathbf{n})}\right|^{\mathbf{2}}}$$

4. LOG-LIKEHOOD-RATIO-BASED DETECTORS

we set out a new framework for the analysis of detection methods based on log-likehood ratio. We extend results of eigendecomposition-based detectors and develop a new test from reflection coefficients. This new test has different advantages : simple implementation, an available costless regularization. From the inverse autocorrelation structure :

$$\frac{|\Phi_n|}{|\Phi_{n-1}|}=\alpha_{n-1}=\frac{|\Omega_{n-1}|}{|\Omega_n|}\Rightarrow|\Omega_n|=\prod_{i=1}^{n-1}\alpha_i^{-1}$$

$$|\Omega_n|=\left(\hat{\sigma}^2\right)^{n-p}.\prod_{i=0}^{p-1}\alpha_i^{-1}=\left(\frac{1}{n-p}\sum_{i=p}^{n-1}\alpha_i^{-1}\right)^{n-p}.\prod_{i=0}^{p}\alpha_i^{-1}$$

$$\frac{\underset{p<q\le n}{Max}\ L(q)}{\underset{q\le p}{Max}L(q)}=\frac{\left[\sum_{i=p}^{n-1}\alpha_i^{-1}/(n-p)\right]^{(n-p).N}}{\left[\prod_{i=p}^{n-1}\alpha_i^{-1}\right]^{N}}=\Lambda(p)$$

$$\begin{cases}\sum_{i=p+1}^{n}\alpha_i^{-1}=\left[\prod_{k=1}^{p}\left(1-|\mu_k|^2\right)\right].P^{(0)}.\left[\sum_{i=p+1}^{n-1}\prod_{k=p+1}^{i}\left(1-|\mu_k|^2\right)\right]\\ \prod_{i=p}^{n-1}\alpha_i^{-1}=\left[\prod_{k=p+1}^{n-1}\left(1-|\mu_k|^2\right)^{n-k}\right].\left[\prod_{k=1}^{p}\left(1-|\mu_k|^2\right)\right]^{n-p}.P^{(0)n-p}\end{cases}$$

$$\mathbf{\Lambda(p)}=\left[\frac{\left[\frac{\mathbf{1}}{\mathbf{n-p}}\sum_{\mathbf{i=p+1}}^{\mathbf{n-1}}\prod_{\mathbf{k=p+1}}^{\mathbf{i}}\left(\mathbf{1}-|\mu_{\mathbf{k}}|^{\mathbf{2}}\right)\right]^{\mathbf{n-p}}}{\prod_{\mathbf{k=p+1}}^{\mathbf{n-1}}\left(\mathbf{1}-|\mu_{\mathbf{k}}|^{\mathbf{2}}\right)^{\mathbf{n-k}}}\right]^{\mathbf{N}}$$

7. CONCLUSION

These new algorithms have different applications in spectrum analysis and array processing when only short data records are available, with real-time implementation advantage thanks to their lattice structure.

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