AN ALGORITHM FOR RECONSTRUCTING POSITIVE IMAGES FROM NOISY DATA

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ABSTRACT

In this paper we describe a novel method for finding non-negative solutions to linear inverse problems. Such problems include image reconstruction where one is required to deconvolve a known point spread function from the image to produce a clearer image. The method described here is related to the truncated singular function expansion for solving linear inverse problems. The method consists of choosing the non-negative solution with minimum 2-norm whose singular function expansion agrees with the truncated singular function expansion solution in its first N terms. The fact that only the first N singular function coefficients, which are easily derived from the data, are used gives the method robustness with respect to noise and the method is not computationally very demanding.

1 Introduction

A recurrent theme in signal processing is the need to analyse data which is linearly related to some unknown function in order to make some deductions about that function. A typical example is where one has a blurred image and a known point spread function and one wishes to deconvolve the two in order to get a clearer image. One often knows in advance that the reconstructed image must be non-negative but many methods do not produce solutions with this property. Another problem is that many methods are very sensitive to noise on the data.

The basic problem we have just described is specified by a linear integral equation of the form

$$g(y) = \int_a^b K(y,x) f(x) dx + noise \tag{1}$$

or in operator formalism

$$g(y) = (Kf)(y) + noise \tag{2}$$

where g represents the data, the kernel K(y,x) is known and the function f is required to be found. In the absence of noise the above equation is known as a Fredholm equation of the first kind. The presence of noise typically makes the determination of f from g an ill-posed linear inverse problem. Essentially this means that there is a wide range of possible solutions corresponding to a given g to within the noise. To solve the problem regularisation is often employed, leading to a smoothed solution which has an inherent resolution associated with it. Detail on a scale finer than this is lost. One example of a regularisation method is the truncated singular function expansion (numerical filtering). In order to restrict the set of possible solutions prior information is often used. One example of this is the knowledge that the solution one is seeking is positive.

2 The Singular Function Approach

The integral operator K is often compact and therefore possesses a singular system $\{\alpha_i; u_i, v_i\}$

$$\left.\begin{array}{l} Ku_i = \alpha_i v_i \\ K^* v_i = \alpha_i u_i \end{array}\right\} \qquad i = 0, ..., \infty \tag{3}$$

where K^* denotes the adjoint of K, the α_i are the singular values and the u_i and v_i are respectively the right and left hand singular functions. The α_i are positive numbers which conventionally decay towards zero with increasing i. If K is a map from $L^2(a,b)$ to $L^2(c,d)$ then the u_i form a basis for $L^2(a,b)$ and the v_i form a basis for $L^2(c,d)$. A standard solution to the problem is then the truncated singular function expansion solution

$$f_{est}(x) = \sum_{i=1}^{N} \frac{b_i}{\alpha_i} u_i(x) \tag{4}$$

where

$$b_i = \int g(y) v_i(y) dy \tag{5}$$

The truncation gives the method its insensitivity to noise since the terms involving the small singular values are excluded. Noise on g gives rise to noisy coefficients b_i and when α_i is small the ratio b_i/α_i will vary greatly depending on the particular realisation of the noise. In practice the data are defined on a discrete set of points rather than lying in L^2. However the data is often a sampled version of a function in L^2 and by solving the inverse problem first with continuous data this might give a clue as to the optimum positioning of the sample points. The truncated singular function expansion solution can be viewed as the solution to the inverse problem of minimum 2-norm whose first N singular function coefficients agree with their "known" values (i.e. as given by (5)). The truncated singular function expansion solution often suffers from the problem that it has negative parts when the true solution is known to be positive.

3 Positive Solutions

Traditionally the method of maximum entropy has been used to try and guarantee positive solutions to linear inverse problems. Various papers by Borwein and Lewis give a detailed analysis of constrained maximum entropy solutions (see, for example, Borwein and Lewis (1991) and the references therein). One can consider searching for a maximum entropy solution whose singular function expansion coincides with the truncated singular function expansion solution in the first N terms. This can be done by extending the approach of Mead and Papanicolaou (1984) for the case when the singular functions form a Chebyshev system. The work of Borwein and Lewis can also be combined with the singular function approach when the singular functions form a Chebyshev system, as they do, for example, for the finite Laplace transform. Such approaches rely on minimising some objective function subject to the solution being positive and obeying several equality constraints - namely the values of the first few coefficients of the singular function expansion. In this paper we are concerned with using the 2-norm as an objective function. The 2-norm has been referred to by some authors as an entropy but we choose not to do this since we believe the term entropy should be restricted to functions of the form $flnf$ or lnf.

Minimising the 2-norm of the solution subject to the equality constraints and the constraint that the function be positive can be viewed as an extension of the truncated singular function expansion solution. Some work, relevant to spectral analysis, has been done on this problem where the constraints involve projections of f onto sines and cosines (see, for example, Borwein and Lewis (1992) Part II).

Returning to the problem in hand we may write it as

$$min\|f\|_2^2 \tag{6}$$

subject to

$$\int f(x)u_i(x)dx = d_i \qquad i = 1, ..., N \tag{7}$$

$$f(x) \geq 0 \tag{8}$$

This is an infinite-dimensional convex programming problem and in the next section we will see how such problems are solved.

4 The Convex Programming Problem

Before dealing with the basic theorem we need the notion of the positive part of a function. For a function s the positive part, s_+ is defined by

$$(s)_+ = s(x), s(x) \geq 0 \tag{9}$$

$$(s)_+ = 0, s(x) < 0 \tag{10}$$

The basic problem is discussed in Michelli et al.(1985). They prove the following theorem
Theorem: Let

$$C = \{g \in L_p(\mu) : g \geq 0\} \tag{11}$$

and

$$D = \{g \in L_p(\mu) : \int_X g\psi_i d\mu = d_i, i = 1, .., n\} \tag{12}$$

and

$$S = C \cap D \tag{13}$$

where $1 < p < \infty$.
Assume

1. (i) S is not empty
2. (ii) There exists a function $\hat{g}$ in S such that the ψ_i are linearly independent over the region for which $\hat{g}$ is strictly greater than zero.

Then there exist real numbers $\{\alpha_i^*\}_{i=1}^n$ such that the element of minimum norm in S is given by

$$f = (\sum_{j=1}^{n} \alpha_j^* \psi_j)_+^{q-1} \tag{14}$$

where

$$1/p + 1/q = 1 \tag{15}$$

We are interested here in the case $p = q = 2$ and where the ψ_i are the singular functions u_i of K. Borwein and Wolkowicz (1986) derive the above solution by using a Lagrange multiplier approach with a constraint qualification different from the usual Slater one. The problem is investigated further in Borwein and Lewis (1992) where a Fenchel duality result for the infinite dimensional case is derived leading to a finite dimensional dual problem for the problem we are interested in. A good discussion of Fenchel duality may be found in Rockafellar (1970). The dual problem can be solved using a Newton method and this is what we have implemented here. The solution to the primal problem is then easily derived from the dual solution. The reasoning proceeds as follows. The dual problem for case $p = q = 2$ is given by

$$\overset{max}{\lambda \in R^n} \; d^T\lambda - 1/2 \int_{t:(A^T\lambda)(t)>0} \sum_{i,j} \lambda_i \lambda_j \psi_i(t)\psi_j(t)dt \tag{16}$$

where

$$(A^T\lambda)(t) = \sum_{i=1}^{N} \lambda_i \psi_i(t) \tag{17}$$

This is equivalent to

$$\min_{\lambda \in R^n} -d^T\lambda + 1/2 \int_{t:(A^T\lambda)(t)>0} \sum_{i,j} \lambda_i \lambda_j \psi_i(t)\psi_j(t)dt \tag{18}$$

Define

$$(H(\lambda))_{ij} = \int_{t:(A^T\lambda)(t)>0} \psi_i(t)\psi_j(t)dt \tag{19}$$

Then we need to minimise

$$-d^T\lambda + 1/2\lambda^T H\lambda \tag{20}$$

Taking the derivative with respect to λ we have, for the gradient

$$g_i = -d_i + (H\lambda)_i \tag{21}$$

The Newton iteration is given by

$$\lambda^{new} = \lambda^{old} - (H^{-1}g)(\lambda^{old}) \tag{22}$$

implying

$$H(\lambda^{old})\lambda^{new} = H(\lambda^{old})\lambda^{old} - (g)(\lambda^{old}) \tag{23}$$

$$= d \tag{24}$$

where the last step is obtained by substituting for g using (21). So, finally, the iteration is

$$H(\lambda^{old})\lambda^{new} = d \tag{25}$$

This then yields on convergence, the optimal value of λ. Let us denote the optimal value of λ by $\bar{\lambda}$. Then the solution to the primal problem is given by

$$\bar{x} = (\sum_i \bar{\lambda}_i \psi_i)^+ \tag{26}$$

5 Examples

The first example is a one-dimensional problem corresponding to one-dimensional coherent band-limited imaging. The kernel in the integral equation governing the inverse problem is the sinc kernel and the corresponding singular functions are the prolate spheroidal wave functions. We have chosen the value c=1.0 for the prolate spheroidal wave functions and in figure 1 the ordinate should be thought of as running from -1 to 1. This value of c is unrealistically low from a band-limited imaging viewpoint but was chosen purely for ease of computation of the prolate spheroidal wave functions. For further details see Bertero and Pike (1982). We assume that the first six singular function coefficients are well known. Though, naturally, we have had to discretise, we have defined the relevant functions on 401 points which gives a good approximation to the continuous problem. The calculation was done using Matlab running under SUNOS4 on a Sun Sparcstation IPX with 48 Mbytes of RAM. The cpu time involved was 3.5 seconds.

The second example is a two-dimensional one corresponding to two-dimensional coherent band-limited imaging with a square object and a square pupil. The kernel in the integral equation is now a product of two sinc functions and the singular functions involved are simply products of the one-dimensional prolate spheroidal wave functions. Again we have chosen c=1.0 for the prolate spheroidal wave functions and the two horizontal axes in figures 2 and 3 can be thought of as running from -1 to 1. For further details see Bertero and Pike (1982). We assume that the first 36 singular function coefficients are well-known. The relevant functions are defined on 401x401 points. The calculation was performed using the same system as for the first example and took approximately 10 minutes of CPU time.

The results for both examples appear to show that this technique has better resolution than the truncated singular function expansion. However one should note that when the truncated singular function expansion is positive this technique will deliver the same result. The improvement in resolution is in contrast to the linear methods for finding positive solutions which have inferior resolution [Bertero et al. 1988].

6 Effects of Noise

With ill-posed linear inverse problems the effect of noise is usually devastating. Use of the truncated singular function expansion makes the solution much more robust with respect to noise. However with the method discussed in this paper there are several questions still to be addressed concerning the effects of noise on the first N singular function coefficients such as

1) When will the problem still have a solution?

2) When such solutions exist what will be the range of solutions corresponding to different noise realisations?

7 Conclusions

We have looked at a method for finding the minimum 2-norm solution to linear inverse problems which is also positive and whose singular function expansion agrees with the truncated singular function expansion solution in its first N terms. If the truncated singular function expansion solution should turn out to be non-negative then clearly our solution will coincide with it. The method is not computationally very expensive and it produces solutions which appear to have better resolution than the truncated singular function expansion solutions, except, of course, when the two types of solution coincide.

8 References

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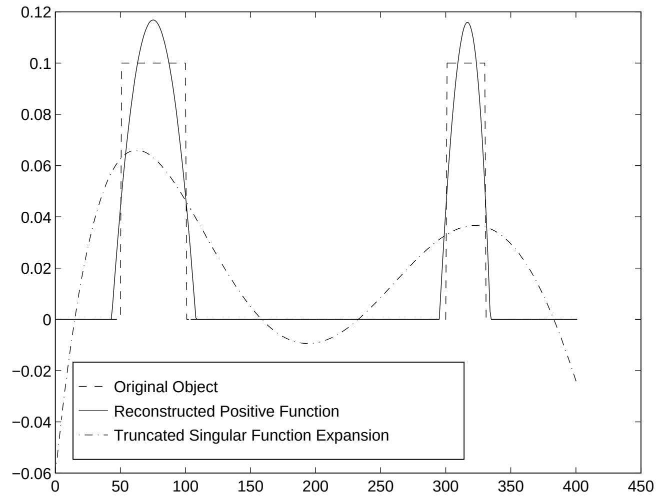

Figure 1: Positive Solution Versus Truncated Singular Function Expansion

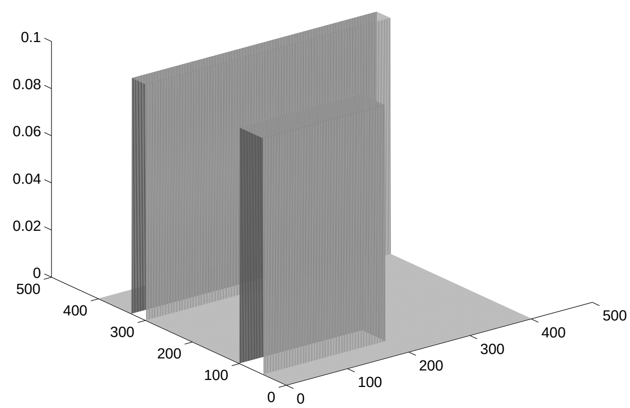

Figure 2: Original object

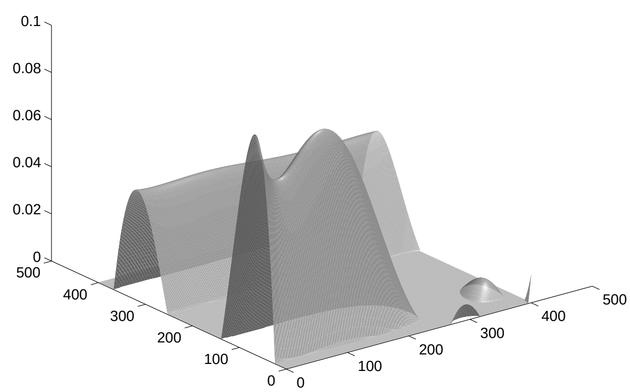

Figure 3: Reconstructed positive function