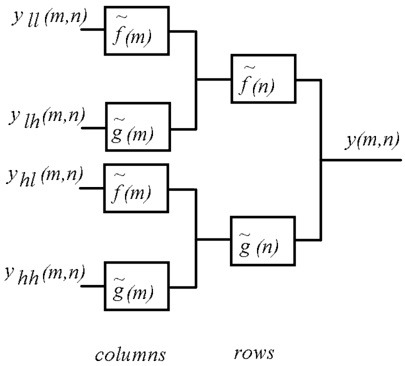

Fig. 2 Four-band image reconstruction

For the shake of the simplicity we consider here the noise free deconvolution. Then our problem is to solve the equation:

$$\mathbf{y} = \mathbf{H}\mathbf{x} \qquad (4)$$

where $\mathbf{y}$ is the vector of available data at full resolution. Let $L_f, L_g, L_{\tilde{f}}, L_{\tilde{g}}$ denotes the length of the causal filters $f, g, \tilde{f}, \tilde{g}$ respectively. In order to work with square blocks Toeplitz matices of square Toeplitz matrices that preserve the commutativity of the convolution operator in matrix notation, both each column and rows of the original image are padded with $L_h - 1 + \max(L_f - 1, L_g - 1)$ zeros before constructing the vector $\mathbf{y}$. However for the shake of the simplicity we do not introduce a new notation. We can decompose $\mathbf{y}$ as follows:

$$\mathbf{y}_{ll} = (\mathbf{F} \otimes \mathbf{F})\mathbf{y} \qquad (5)$$

$$\mathbf{y}_{lh} = (\mathbf{F} \otimes \mathbf{G})\mathbf{y} \qquad (6)$$

$$\mathbf{y}_{hl} = (\mathbf{G} \otimes \mathbf{F})\mathbf{y} \qquad (7)$$

$$\mathbf{y}_{hh} = (\mathbf{G} \otimes \mathbf{G})\mathbf{y} \qquad (8)$$

$\mathbf{F}, \mathbf{G}$ are Toeplitz matrices constructed with the filter impulse response *f(n)* and *g(n)* respectevely. The symbol $\otimes$ denotes the Kronecker product.
It can be noted that the matrices $\mathbf{F} \otimes \mathbf{F}$, $\mathbf{F} \otimes \mathbf{G}$, $\mathbf{F} \otimes \mathbf{G}$, $\mathbf{G} \otimes \mathbf{G}$ are block Toeplitz matrices.
By incorporating (4) in (5),(6),(7),(8) and using the commutativity of the block Toeplitz matrices it follows that:

$$\mathbf{y}_{ll} = \mathbf{H}\mathbf{x}_{ll}, \mathbf{y}_{lh} = \mathbf{H}\mathbf{x}_{lh}, \mathbf{y}_{hl} = \mathbf{H}\mathbf{x}_{hl}, \mathbf{y}_{hh} = \mathbf{H}\mathbf{x}_{hh} \qquad (9)$$

with

$$\mathbf{x}_{ll} = (\mathbf{F} \otimes \mathbf{F})\mathbf{x}, \mathbf{x}_{lh} = (\mathbf{F} \otimes \mathbf{G})\mathbf{x},$$

$$\mathbf{x}_{hl} = (\mathbf{G} \otimes \mathbf{F})\mathbf{x}, \mathbf{x}_{hh} = (\mathbf{G} \otimes \mathbf{G})\mathbf{x} \quad (10)$$

$\mathbf{x}_{ll}$ is the image approximation, $\mathbf{x}_{lh}$, $\mathbf{x}_{hl}$ and $\mathbf{x}_{hh}$ denotes the vertical detail-image, horizontal detail-image and the diagonal detail-image, respectively.
By solving the four equations (9) we obtain the approximation and the details of the solution $\mathbf{x}$ at the resolution 2^{-1}. Then we reconstruct the full-band solution using the reconstruction scheme of Fig. 2. In matrix notation we have:

$$\tilde{\mathbf{x}} = (\tilde{\mathbf{F}} \otimes \tilde{\mathbf{F}})\mathbf{x}_{ll} + (\tilde{\mathbf{G}} \otimes \tilde{\mathbf{F}})\mathbf{x}_{lh} + (\tilde{\mathbf{F}} \otimes \tilde{\mathbf{G}})\mathbf{x}_{hl} + (\tilde{\mathbf{G}} \otimes \tilde{\mathbf{G}})\mathbf{x}_{hh} \ (11)$$

$\tilde{\mathbf{F}}, \tilde{\mathbf{G}}$ are Toeplitz matrices constructed with the filter impulse response $\tilde{f}(n)$ and $\tilde{g}(n)$, respectively.

3. Regularized Deconvolution in a Multiresolution Scheme.

When data are noisy the available data in a multiresolution scheme becomes (see (2), (9)):

$$\mathbf{y}_u = \mathbf{H}_u + \mathbf{n}_u \qquad (12)$$

u denotes ll or lh or hl or hh.
We propose to use the Miller regularization method [1] to solve separatly the four equations in (12). The following constraints about the solutions $\hat{\mathbf{x}}_u$ are used :

$$\|\mathbf{y}_u - \mathbf{H}\hat{\mathbf{x}}_u\|^2 \le \|\mathbf{n}_u\|^2 \ (13)$$

$$\|\mathbf{B}_u\hat{\mathbf{x}}_u\|^2 \le r_u^2 \qquad (14)$$

where $\mathbf{B}_u$ is usually a high-pass filter and r_u^2 is the energy of the filtered approximation or detail solutions according to u. The energy of the filtered approximation and detail images are a measure of the regularity of the solutions into frequency bands. Following the Miller approach the constraints are quadratically combined as :

$$\|\mathbf{y}_u - \mathbf{H}\hat{\mathbf{x}}_u\|^2 + \frac{\|\mathbf{n}_u\|^2}{r_u^2}\|\mathbf{B}_u\hat{\mathbf{x}}_u\|^2 \le 2\|\mathbf{n}_u\|^2 \quad (15)$$

Then the four problems are the solutions of the normal equations :

$$(\mathbf{H}^T\mathbf{H} + \alpha_u\mathbf{B}_u^T\mathbf{B}_u)\hat{\mathbf{x}}_u = \mathbf{H}^T\mathbf{y} \quad (16)$$

where

$$\alpha_u = \frac{\|\mathbf{n}_u\|^2}{r_u^2} \qquad (17)$$

In practice regularity coefficients r_u^2 and noise energies $\|n_u\|^2$ are unknown. Fortunatelly these parameters can be estimated using cross-validation or generalized cross-validation [5].
Solving equations (16) requires the evaluation of the inverse of matrices:

$$\mathbf{A}_u = \mathbf{H}^T\mathbf{H} + \alpha_u\mathbf{B}_u^T\mathbf{B}_u \quad (18)$$

The quality of the solutions $\hat{\mathbf{x}}_u$ depend on the condition number of the matrices A_u.
We denote $\lambda_{\mathbf{H}}^2, \lambda_{\mathbf{B}_u}^2$ as the singular values of $\mathbf{H}, \mathbf{B}_u$, respectively. Then the condition number of (18) is:

$$C[A_u] = \frac{[\lambda_H^2 + \alpha_u\lambda_{B_u}^2]_{max}}{[\lambda_H^2 + \alpha_u\lambda_{B_u}^2]_{min}} \qquad (19)$$

Clearly the operators $\mathbf{B}_u$ should be chosen with large singular values (resp small singular values) when singular values of $\mathbf{H}$ are small (resp large).
However solving (16) by left multiplication by the inverse of $\mathbf{A}_u$ leads to a reconstructed solution given by (11) which does not respect some properties of the signal such as positivity or support. In order to circumvent these artifacts we propose to solve (16) using iterative constrained algorithms. A general formulation of this problem is given in [6].