Discrete B–spline Functions

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ABSTRACT

A simple discrete version of B–splines is proposed. The proposed discrete version has different values from B–splines at the discrete points, but it is proven that the proposed discrete version tends to B–splines when the sampling interval goes to zero. They can be evaluated more quickly than the former discrete B–splines, only by RRS digital filters.

1 INTRODUCTION

B–splines have the property that they can be generated as a multifold convolution integral of rectangular functions [1]. This property leads to an analog circuit [2] to generate B–splines in the continuous domain. Another important property is that the smoothness of their wave forms in the sense of differentiability is variable according to their order m. This property is advantageous in approximating curves, curved surfaces, and solutions for differential equations.

Since it is a discrete domain where data to be approximated are available and values of an approximate function can be evaluated in the computers, functions defined in the same discrete domain as the data are more compatible with the computer implementation than those defined in a continuous domain. Hence, discrete B–splines are proposed [3],[4] which have the same value as B–splines at the discrete points equidividing the knot interval. The fast algorithm to compute discrete B–splines has been also presented [3],[4] based on their representation as the discrete convolution of the sampled rectangular functions and a sampled B–spline. Therefore, they can be evaluated by RRS digital filters [5] and a short FIR digital filter.

To substitute B–splines in interpolation, their discrete version must have the same sampled values at discrete points. It is nice for approximation, too, because its staircase interpolation is automatically a good approximation of B–splines in $L^2(-\infty,\infty)$. However, it is not mandatory to have the same values in approximation. But it is crucial for a substitute to be a good approximation of B–splines.

Emphasizing the analogy in the definition by means of convolution, we can consider another discrete version of B–splines. In this paper, we propose a simpler discrete version of B–splines, named Discrete B–spline functions, simply as the multifold discrete version of the sampled rectangular functions.

This version can be evaluated more quickly than the former discrete B–splines. They have different values from B–splines at the discrete points, but it is proven that the proposed discrete version tends to B–splines when the sampling interval goes to zero.

2 PRELIMINARIES

B–splines are given [1] as

$$\beta^m(t) \quad := \quad m\sum_{i=0}^{m}\frac{(-1)^i}{(m-i)!i!}(t-i)_+^{m-1}, \tag{1}$$

where m is called the order of B–splines. And it has been proven [1] that

$$\beta^m(t)=0, \qquad t<0 \;\text{ or }\; t\geq m. \tag{2}$$

Discrete B–splines are given [3],[4] as

$$\begin{aligned} b_N^m\left(\frac{k}{N}\right) \quad &:= \quad \beta^m\left(\frac{k}{N}\right) \\ &= \quad \frac{1}{N^{m-1}}\{(\underbrace{b_N^1 * b_N^1 * \cdots * b_N^1}_{m}) * b_1^m\}(k), \end{aligned} \tag{3}$$

where N is a positive integer but one.

3 PROPOSAL OF DISCRETE B–SPLINE FUNCTIONS

We define the proposed discrete version as a multifold discrete convolution of sampled rectangular functions. The explicit formula of the proposed discrete version is given in this section.

Lemma 3.1 *The $(m-1)$–fold convolution $x_N^m(k)$ of the sampled rectangular function $b_N^1(k)$ can be written as*

$$x_N^m(k) \quad := \quad (\underbrace{b_N^1 * b_N^1 * \cdots * b_N^1}_{m})(k)$$

$$= \begin{cases} b_N^1(k), & m=1, \\ m \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \prod_{j=1}^{m-1} (k-Ni+j)_+, & \\ & m=2,3,\dots, \end{cases} \tag{4}$$

$$\textit{where} \qquad (a)_+ = \begin{cases} a, & a \ge 0, \\ 0, & a < 0. \end{cases} \tag{5}$$

(Proof) Apparently (4) is true in the case m=1. In the case m=2,

$$\begin{aligned} & x_N^2(k) \\ &= (b_N^1 * b_N^1)(k) \\ &= \sum_{\ell=-\infty}^{\infty} b_N^1(\ell) b_N^1(k-\ell) \\ &= \sum_{\ell=0}^{N-1} b_N^1(k-\ell) \\ &= \sum_{\ell=k-N+1}^{k} b_N^1(\ell) \\ &= \begin{cases} k+1, & k=0,1,\cdots,N-1 \\ 2N-k-1, & k=N,N+1,\cdots,2N-1 \\ 0, & \text{otherwise} \end{cases} \\ &= (k+1)_+ - 2(k-N+1)_+ + (k-2N+1)_+ \\ &= 2\sum_{i=0}^{2} \frac{(-1)^i}{(2-i)!i!}(k-Ni+1)_+, \end{aligned} \tag{6}$$

which means that (4) holds good for $m = 2$. Assume that (4) holds good for $m \ge 2$, *i.e.*,

$$x_N^m(k) = m \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \prod_{j=1}^{m-1} (k-Ni+j)_+. \tag{7}$$

Then we have

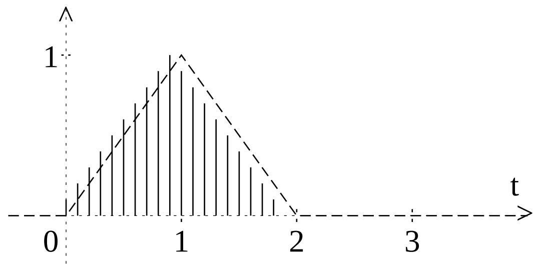
$$\begin{aligned} & x_N^{m+1}(k) \\ &= (\underbrace{b_N^1 * b_N^1 * \cdots * b_N^1}_{m+1})(k) \\ &= (b_N^1 * x_N^m)(k) \\ &= \sum_{\ell=-\infty}^{\infty} b_N^1(\ell) x_N^m(k-\ell) \\ &= \sum_{\ell=0}^{N-1} x_N^m(k-\ell) \\ &= \sum_{\ell=k-N+1}^{k} x_N^m(\ell) \\ &= \sum_{\ell=k-N+1}^{k} m \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \prod_{j=1}^{m-1} (\ell-Ni+j)_+ \\ &= m \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \sum_{\ell=k-N+1}^{k} \prod_{j=1}^{m-1} (\ell-Ni+j)_+ \\ &= m \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \sum_{\ell=k-N+1}^{k} \frac{1}{m} \left\{ \prod_{j=1}^{m} (\ell-Ni+j)_+ \right. \\ &\qquad \left. - \prod_{j=1}^{m} (\ell-1-Ni+j)_+ \right\} \\ &= \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \left\{ \prod_{j=1}^{m} (k-Ni+j)_+ \right. \\ &\qquad \left. - \prod_{j=1}^{m} (k-N(i+1)+j)_+ \right\} \\ &= \frac{1}{m!0!} \prod_{j=1}^{m} (k+j)_+ + \sum_{i=1}^{m} \frac{(-1)^i}{(m-i)!i!} \prod_{j=1}^{m} (k-Ni+j)_+ \\ &\qquad - \sum_{i=1}^{m} \frac{(-1)^i}{(m-i)!i!} \prod_{j=1}^{m} (k-N(i+1)+j)_+ \\ &\qquad - \frac{(-1)^m}{0!m!} \prod_{j=1}^{m} (k-N(m+1)+j)_+ \\ &= \frac{m+1}{(m+1)!0!} \prod_{j=1}^{m} (k+j)_+ \\ &\qquad + \sum_{i=1}^{m} \left\{ \frac{(-1)^i}{(m-i)!i!} - \frac{(-1)^{i-1}}{(m-i+1)!(i-1)!} \right\} \\ &\qquad \times \prod_{j=1}^{m} (k-Ni+j)_+ \\ &\qquad + \frac{(-1)^{m+1}(m+1)}{0!(m+1)!} \prod_{j=1}^{m} (k-N(m+1)+j)_+ \\ &= \frac{m+1}{(m+1)!0!} \prod_{j=1}^{m} (k+j)_+ \\ &\qquad + \sum_{i=1}^{m} \frac{(-1)^i(m+1)}{(m+1-i)!i!} \prod_{j=1}^{m} (k-Ni+j)_+ \\ &\qquad + \frac{(-1)^{m+1}(m+1)}{0!(m+1)!} \prod_{j=1}^{m} (k-N(m+1)+j)_+ \\ &= (m+1) \sum_{i=0}^{m+1} \frac{(-1)^i}{(m+1-i)!i!} \prod_{j=1}^{m} (k-Ni+j)_+, \end{aligned} \tag{8}$$

which means that (4) holds good for the case $m+1$. The above (6)–(8) and the mathematical induction completes a proof of Lemma 3.1. □

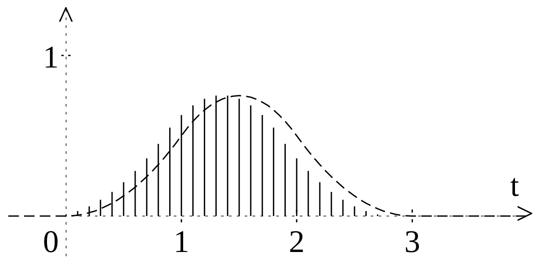
Define the proposed discrete version $y_N^m\left(\frac{k}{N}\right)$ as

$$y_N^m\left(\frac{k}{N}\right) := \frac{1}{N^{m-1}} x_N^m(k). \tag{9}$$

Figure 1 shows some examples of the proposed discrete version. The broken lines in Fig.1 show B–splines.



(i) In the case m=2 and N=10



(ii) In the case m=3 and N=10

Figure 1: Examples of Discrete B–spline functions

4 LIMIT PROPERTY OF DISCRETE B–SPLINE FUNCTIONS

In this section, it shall be proven that the proposed discrete version satisfies that the distance between the discrete version and the original becomes zero at the limit $N \to \infty$.

The staircase interpolation $\tilde{y}_N^m(t)$ of the discrete version $y_N^m\left(\frac{k}{N}\right)$ defined by

$$\tilde{y}_N^m(t) = y_N^m\left(\frac{k}{N}\right), \qquad \frac{k}{N} \le t < \frac{k+1}{N}, \tag{10}$$

can be written as

$$\tilde{y}_N^m(t) = \begin{cases} \beta^1(t), & m=1, \\ \dfrac{m}{N^{m-1}} \displaystyle\sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \prod_{j=1}^{m-1} (\lceil tN \rceil - Ni + j)_+, & m = 2, 3, \cdots, \end{cases} \tag{11}$$

by Lemma 3.1, where $\lceil a \rceil$ is a maximum integer not exceeding a.

Lemma 4.1

$$\tilde{y}_N^m(t) = 0, \qquad t<0 \ \ or \ \ t \ge m. \tag{12}$$

(Proof) Clearly (12) is true in the case m=1 since

$$\tilde{y}_N^1(t) = \beta^1(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Assume that (12) holds good for $m \ge 1$, *i.e.*,

$$\tilde{y}_N^m(t) = 0, \qquad t<0 \ \text{ or } \ t \ge m. \tag{14}$$

Then we have

$$\begin{aligned} \tilde{y}_N^{m+1}(t) &= y_N^{m+1}\left(\frac{\lceil tN \rceil}{N}\right) \\ &= \frac{1}{N^m} x_N^{m+1}(\lceil tN \rceil) \\ &= \frac{1}{N^m} \left(b_N^1 * x_N^m\right)(\lceil tN \rceil) \\ &= \frac{1}{N^m} \sum_{\ell=-\infty}^{\infty} b_N^1(\ell) x_N^m(\lceil tN \rceil - \ell) \\ &= \frac{1}{N^m} \sum_{\ell=0}^{N-1} x_N^m(\lceil tN \rceil - \ell) \\ &= \frac{1}{N} \sum_{\ell=0}^{N-1} \tilde{y}_N^m\left(t - \frac{\ell}{N}\right) \\ &= 0, \qquad t<0 \ \text{ or } \ t \ge m+1, \end{aligned} \tag{15}$$

which means (12) holds good for the case $m+1$. The above (13)–(15) and the mathematical induction completes a proof of Lemma 4.1. □

From (1)–(12), we have the following theorem.

Theorem 4.1

$$\|\tilde{y}_N^m - \beta^m\|^2_{L^2(-\infty,\infty)} := \int_{-\infty}^{\infty} \left|\tilde{y}_N^m(t) - \beta^m(t)\right|^2 dt \to 0, \quad (N \to \infty). \tag{16}$$

(Proof) From (13), obviously it is true for any $N \ge 2$ in the case $m=1$. In the case $m \ge 2$,

$$\begin{aligned} &\|\tilde{y}_N^m - \beta^m\|^2_{L^2(-\infty,\infty)} \\ &= \int_{-\infty}^{\infty} \left|\tilde{y}_N^m(t) - \beta^m(t)\right|^2 dt \\ &= \int_0^m \left|\tilde{y}_N^m(t) - \beta^m(t)\right|^2 dt, \\ &\qquad \text{(by (12) of Lemma 3.1 and (2))} \\ &= \int_0^m \left| \frac{m}{N^{m-1}} \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \prod_{j=1}^{m-1} (\lceil tN \rceil - Ni + j)_+ \right. \\ &\qquad \left. - m \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} (t-i)_+^{m-1} \right|^2 dt, \\ &\qquad \text{(by (1) and (11))} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^m \left| m \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!i!} \left\{ \frac{1}{N^{m-1}} \right.\right. \\
&\qquad \left.\left. \times \prod_{j=1}^{m-1} (\lceil tN \rceil - Ni + j)_+ - (t-i)_+^{m-1} \right\} \right|^2 dt \\
&\le \int_0^m \left| m \sum_{i=0}^{m} \left\{ \frac{1}{N^{m-1}} \prod_{j=1}^{m-1} (\lceil tN \rceil - Ni + j)_+ \right.\right. \\
&\qquad \left.\left. -(t-i)_+^{m-1} \right\} \right|^2 dt \\
&\le \int_0^m \left| m \sum_{i=0}^{m} \left\{ \frac{1}{N^{m-1}} \prod_{j=1}^{m-1} (\lceil tN \rceil + j)_+ - t_+^{m-1} \right\} \right|^2 dt \\
&= \int_0^m \left| m(m+1) \left\{ \frac{1}{N^{m-1}} \right.\right. \\
&\qquad \left.\left. \times \prod_{j=1}^{m-1} (\lceil tN \rceil + j)_+ - t_+^{m-1} \right\} \right|^2 dt \\
&= m^2(m+1)^2 \\
&\qquad \times \int_0^m \left| \left\{ \frac{1}{N^{m-1}} \prod_{j=1}^{m-1} (\lceil tN \rceil + j) - t^{m-1} \right\} \right|^2 dt \\
&\le m^2(m+1)^2 \int_0^m \left| \frac{1}{N^{m-1}} \right. \\
&\qquad \left. \times \left\{ \prod_{j=1}^{m-1} (\lceil tN \rceil + j) - \lceil tN \rceil^{m-1} \right\} \right|^2 dt \\
&= m^2(m+1)^2 \sum_{k=0}^{mN-1} \\
&\qquad \frac{1}{N} \left| \frac{1}{N^{m-1}} \left\{ \prod_{j=1}^{m-1} (k+j) - k^{m-1} \right\} \right|^2 \\
&\le m^2(m+1)^2 \frac{mN}{N} \\
&\qquad \times \left| \frac{1}{N^{m-1}} \left\{ \prod_{j=1}^{m-1} (mN+j) - (mN)^{m-1} \right\} \right|^2 \\
&= m^3(m+1)^2 \left| \prod_{j=1}^{m-1} \left(m + \frac{j}{N} \right) - m^{m-1} \right|^2 \\
&= m^3(m+1)^2 \left(\sum_{\ell=1}^{m-1} \frac{c_\ell}{N^\ell} m^{m-\ell-1} \right)^2 \to 0, \quad (N \to \infty),
\end{aligned}
\tag{17}
$$

where $\{c_\ell\}_{\ell=1}^{m-1}$ means the coefficients of $m^{m-\ell-1}$ of a polynomial in m. This completes a proof of Theorem 4.1. □

Theorem 4.1 guarantees that the discrete version becomes as close as required to the original B–spline functions by making the sampling interval $1/N$ shorter while it keeps the domain compatible with the discrete data.

5 CONCLUSIONS

Discrete B–spline functions were proposed as a simple discrete version of B–splines. The proposed discrete version can be evaluated more rapidly than discrete B–splines in approximation of data such as free curves in computer graphics.

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