Generalized time-frequency distributions and applications

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Abstract A decomposition of the derivatives of order ℓ of a polynomial $\phi(t)$ is proposed in terms of $\phi(t-t_0), \dots, \phi(t-t_n)$. This result allows us to introduce generalized time-frequency distributions for studying polynomial phase signals with constant amplitude in order to determine the degree and the coefficients of the corresponding phase. Relationships between these distributions and the already known polynomial distributions are established. Statistical properties of the proposed distributions are studied and their application for estimating the instantaneous frequencies in multiple chirp signals are discussed.

1 INTRODUCTION

Detection and estimation of polynomial phase signals are important problems in the engineering applications, for instance, in the radar processing. Many authors elaborate techniques for studying these problems. New tools for the estimation of such signals have been introduced, for instance : the Polynomial phase Transform (PT) [1] [2], the Polynomial Wigner-Ville distribution (PWVD) [3] [4], the Generalized Ambiguity Function (GAF) [5] [6] . . . In this paper, we first recall some properties of the GAF already introduced in [5]. Statistical properties of the GAF are studied in some particular cases. Then an algorithm for the estimation of polynomial phase signals is proposed. Finally it is shown that this algorithm is similar to the one proposed in [1]. A GAF based method is proposed for estimating the instantaneous frequencies in multiple chirp signals.

2 MAIN RESULTS

The classical symmetric polynomials of the parameters $x_1, ..., x_n$ are denoted by :

$$\sigma_n^p(x_1, ..., x_n) \triangleq \sum_{(i_1, i_2, ..., i_p) \in E_p} x_{i_1} x_{i_2} \dots x_{i_p} \tag{1}$$

where E_p denotes the set of all the parts of $\{1, 2, \dots, n\}$ having p elements and $p = 0, \dots, n$. The following notation can also be used

$$\sigma_{n-1}^p(x_{l \neq k}) \triangleq \sigma_{n-1}^p(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n). \tag{2}$$

Proposition 1 *Let $t_0, ..., t_Q$ denote $Q+1$ arbitrary distinct numbers, $\mathcal{R}_N[t]$ the set of all the polynomials of degree $\leq N$ and consider the following identity :*

$$\tau^\ell \phi^{(\ell)}(t) = \sum_{k=0}^{Q} \alpha_k^\ell \phi(t - \tau t_k), \quad \forall \tau, \forall t, \forall \phi \in \mathcal{R}_N[t] \tag{3}$$

where $\phi^{(\ell)}$ denotes the derivative of order $\ell = 0, \dots, N$. Then the following properties hold.
1. For $N \leq Q$, the α_k^ℓ's always exist and they are unique for $N = Q$
2. For $N = Q+1$ and $\ell \leq Q$, the α_k^ℓ's exist iff the t_k's satisfy :

$$\sigma_{Q+1}^{Q+1-\ell}(t_0, ..., t_Q) = 0. \tag{4}$$

In both cases, the α_k^ℓ's are independent of the polynomial ϕ and can be expressed as :

$$\alpha_k^\ell = \ell! \frac{\sigma_Q^{Q-\ell}(t_{l \neq k})}{\prod_{i \neq k}(t_i - t_k)}, \quad k = 0, ..., Q. \tag{5}$$

2.1 Generalized time-frequency distributions

The result of the proposition above is important and allows us to introduce nonlinear operators for studying polynomial phase signals with a constant amplitude.

Definition 1 *Let $t_0, ..., t_Q$ denote $Q+1$ arbitrary distinct numbers and $\alpha_0^\ell, \dots, \alpha_Q^\ell$ the corresponding parameters given by (5). The Generalized Ambiguity Function (GAF) of order $Q+1$, computed for a signal $z(t)$ and denoted by $\mathcal{A}_{Q+1}^\ell(z, \nu, \tau)$, is defined by :*

$$\mathcal{A}_{Q+1}^\ell(z, \nu, \tau) = \int_{-\infty}^{\infty} \prod_{k=0}^{Q} z(t - t_k \tau)^{\alpha_k^\ell} e^{-j2\pi\nu t} dt. \tag{6}$$

The Generalized Wigner Distribution (GWD) of order $Q+1$ denoted by $\mathcal{W}_{Q+1}^\ell(z, \nu, t)$ is defined by :

$$\mathcal{W}_{Q+1}^\ell(z, \nu, t) = \int_{-\infty}^{\infty} \prod_{k=0}^{Q} z(t - t_k \tau)^{\alpha_k^\ell} e^{-j2\pi\nu\tau^\ell} \tau^{\ell-1} d\tau. \tag{7}$$

The order is the number of the parameters t_k appearing in the definition of the operator.

The main results of the GAF and the GWD are summarized in the following propositions.

Proposition 2 *Let $\mathcal{A}^{\ell}_{Q+1}$ and $\mathcal{W}^{\ell}_{Q+1}$ be the GAF and the GWD constructed with the parameters $t_0, \dots t_Q$ and consider the polynomial phase signal :*

$$z(t) = A\,\mathrm{rect}(\frac{t}{T})e^{j2\pi\phi(t)}, \quad \phi(t) \triangleq \sum_{i=0}^{N} a_i t^i \tag{8}$$

where ϕ is a polynomial whose degree satisfies $d^o(\phi) \leq N$ and N a given integer. Then the following properties hold.

1. For $Q \geq N$, one has

$$\mathcal{A}^{\ell}_{Q+1}(z,\nu,\tau) = A^{\delta(\ell)} \int_0^T e^{-j2\pi[\nu t - \tau^{\ell}\phi^{(\ell)}(t)]}dt \tag{9}$$

where $\delta(\ell) = 1$ for $\ell = 0$ and 0 otherwise. The global maximum with respect to ν of $|\mathcal{G}^{N-1}_{Q+1}(z,\nu,\tau)|$ equals T and is obtained for $\nu = N!\tau^{N-1}a_N$, i.e.,
(i) if $d^o(\phi) < N$, then $a_i = 0$, for $i = d^o(\phi)+1, \dots, N$.
(ii) if $d^o(\phi) = N$, the coefficient a_N is given by :

$$a_N = \frac{1}{N!\tau^{N-1}} \arg\max_{\nu} |\mathcal{A}^{N-1}_{Q+1}(z,\nu,\tau)|. \tag{10}$$

2. For $Q = N-1$, the result above holds if the t_k's satisfy (4), i.e., $t_0 + \dots + t_Q = 0$.
3. The coefficient a_N is also given by :

$$a_N = \frac{1}{N!\tau^{N-1}} \frac{\int_{-\infty}^{\infty} \nu \mathcal{A}^{N-1}_{Q+1}(z,\nu,\tau)d\nu}{\int_{-\infty}^{\infty} \mathcal{A}^{N-1}_{Q+1}(z,\nu,\tau)d\nu}. \tag{11}$$

4. For $Q \geq N$, one has :

$$\mathcal{W}^{\ell}_{Q+1}(z,\nu,t) = \frac{1}{\ell}\delta(\nu - \phi^{(\ell)}(t)) \quad \forall \ell \neq 0. \tag{12}$$

2.2 Relationship between the GAF and the PT

The PT introduced in [1] in order to estimate the coefficients of a polynomial phase is defined as

$$PT_N(z,\nu,\tau) = \int_0^T \prod_{q=0}^{N-1} \left(z^{\$q}(t-q\tau)\right)^{C^q_{N-1}} e^{-j2\pi\nu t}dt. \tag{13}$$

The main property of the PT can be stated as follows. For polynomial phase signals given by (8), the global maximum of $|PT_N(z,\nu,\tau)|$ is obtained for $\nu = N!a_N\tau^{N-1}$, i.e.,

$$a_N = \frac{1}{N!\tau^{N-1}} \arg\max_{\nu} |PT_N(z,\nu,\tau)|. \tag{14}$$

Thus the distributions $\mathcal{A}^{N-1}_{Q+1}(z,\nu,\tau)$ and $PT_N(z,\nu,\tau)$ are very close and the following result gives the relationship between these two operators.

Proposition 3 *If $z(t)$ is a complex signal defined by (8), one has :*

$$\mathcal{A}^{N-1}_{Q+1} = A^{\delta(N-1)-2^{N-1}} e^{j\pi N!(N-1)a_N\tau^N} PT_N.$$

This relationship gives an important theoretical result. In particular, it shows that the maxima of the operators PT_N and $\mathcal{A}^{N-1}_{Q+1}$ are obtained for the same value of the argument. However, these two operators are different in their conception. Indeed, according to its definition, the GAF can be constructed by choosing arbitrarily the parameters t_k and the definition of the PT_N is unique and does not introduce any arbitrary parameter. Even if the two operators give the same estimation of the polynomial phase coefficient, they can have different performances in presence of an additive noise.
Consider the particular case $Q = \ell = N-1$, i.e,

$$\mathcal{A}^{N-1}_N(z,\nu,\tau) = \int_{-\infty}^{\infty} \prod_{k=0}^{N-1} z(t-t_k\tau)^{\alpha^{N-1}_k} e^{-j2\pi\nu t}dt \tag{15}$$

where the t_k's satisfy $t_0 + \dots + t_Q = 0$ and introduce the simplified notation :

$$\mathcal{A}_N(z,\nu,\tau) \triangleq \mathcal{A}^{N-1}_N(z,\nu,\tau).$$

Taking the particular values $t_k = k - \frac{N-1}{2}$, $k = 0, ..., N-1$, one obtains :

$$\mathcal{A}_N(z,\nu,\tau) = \int_{-\infty}^{\infty} \prod_{k=0}^{N-1} \left(z^{\$k}(t-t_k\tau)\right)^{C^k_{N-1}} e^{-j2\pi\nu t}dt. \tag{16}$$

For instance, one obtains for $N = 0, 1$:

$$\mathcal{A}_1(z,\nu,\tau) = \int_{-\infty}^{\infty} z(t)e^{-j2\pi\nu t}dt$$

$$\mathcal{A}_2(z,\nu,\tau) = \int_{-\infty}^{\infty} z(t+\tau/2)z^*(t-\tau/2)e^{-j2\pi\nu t}dt$$

The following proposition gives us the relationship between this particular version of the GAF and the PT.

Proposition 4 *If $z(t)$ is a complex signal defined by (8), one has :*

$$\mathcal{A}_N(z,\nu,\tau) = PT_N(z,\nu,\tau)e^{j\pi N!(N-1)a_N\tau^N}$$

and thus

$$|\mathcal{A}_N(z,\nu,\tau)| = |PT_N(z,\nu,\tau)|.$$

2.3 Relationship between the GAF and the PWVD

The PWVD has been introduced in [3] in order to ensure the concentration of the energy of a signal around its instantaneous frequency. This distribution is defined by

$$PW_Q(z,t,v) \triangleq \int_{-\infty}^{\infty} \prod_{k=0}^{Q-1} z(t-t_k\tau)^{\alpha^1_k} e^{-j2\pi\nu\tau}d\tau \tag{17}$$

The relationship between the GAF and the PWVD is :

$$\mathcal{A}^1_Q(z,\nu,\tau) = \mathcal{F}_{t\to\nu}\mathcal{F}^{-1}_{v\to\tau}\left[PW_Q(z,t,v)\right]. \tag{18}$$

2.4 Statistical analysis of $\mathcal{A}_N(z,\nu,\tau)$

Let us introduce the discrete version of the GAF. Denoting by $\underline{\mathcal{A}}_N$ the operator associated to $\mathcal{A}_N$ in the discrete case, by T_e the sampling period and by N_e the number of samples, one obtains :

$$\underline{\mathcal{A}}_N(z,\nu,\tau) = \sum_{n=n^-}^{n^+} \prod_{k=0}^{N-1} \left(z^{\$k}(nT_e - t_k\tau T_e)\right)^{c_k} e^{-j2\pi\nu nT_e} \tag{19}$$

where $n^- = 1 + \frac{N-1}{2}$, $n^+ = N_e - \frac{N-1}{2}$ and $c_k \triangleq C^k_{N-1}$. In this discrete version, the parameter τ is assumed to be positive and to satisfy :

$$\tau \leq \frac{N_e - 1}{N-1}. \tag{20}$$

Let z_k be a discrete signal affected by an additive zero-mean complex circular Gaussian white noise w_k with variance σ_w^2. The resulting signal y_k is given by :

$$y_k = z_k + w_k = Ae^{j2\pi\phi(kT_e)} + w_k, \quad 1 \leq k \leq N_e. \tag{21}$$

The following results give the first- and second-order moment of $\underline{\mathcal{A}}_N(y,\nu,\tau)$.

Proposition 5 *The operator $\underline{\mathcal{A}}_N$ is unbiased for any value of ν and τ, i.e., one has*

$$E\left[\underline{\mathcal{A}}_N(y,\nu,\tau)\right] = \underline{\mathcal{A}}_N(z,\nu,\tau). \tag{22}$$

Proposition 6 *It is always possible to choose τ in order to ensure the independency between the random variables $y(n - t_k\tau)$ and $y(m - t_l\tau)$. For such values of τ, the second-order moment can be expressed as follows :*

$$E\left[|\underline{\mathcal{A}}_N(y,\nu,\tau)|^2\right] = |\underline{\mathcal{A}}_N(z,\nu,\tau)|^2 + A^{2^N}(N_e - (N-1)\tau)K$$

where

$$K \triangleq \prod_{k=0}^{N-1}\left[\sum_{i=0}^{C^k_{N-1}} (C^i_{C^k_{N-1}})^2 i!(\frac{1}{SNR})^i\right] - 1 \tag{23}$$

It is worth noting that the result above is established in a general case and for $\tau = \frac{N_e}{N}$, one obtains the expression proposed in [1].

Proposition 7 *Let $\underline{P}(z,\nu,\tau)$ be the discrete version of the PT, $\tau = \frac{N_e}{N}$ and $t_k = k - \frac{N-1}{2}$, $k = 0,..,N-1$. Then*

$$\underline{\mathcal{A}}_N(z,\nu,\tau) = e^{j\pi(N-1)\tau\nu}\underline{P}(z,\nu,\tau) \tag{24}$$

According to the result above, the particular form of the GAF would have the same performances as the PT when $\tau = \frac{N_e}{N}$. However it is interesting to study the performances of the GAF in its general form. In the rest of the paper, we restrict our study to some applications of the GAF.

3 APPLICATIONS

In this section, we consider the case $Q = \ell = N-1$. According to Proposition 1, one has to assume that the t_k's satisfy $t_0 + \ldots + t_Q = 0$.

3.1 Estimation of the coefficients in the presence of additive noise

The results of the previous section allow us to propose a sequential algorithm for the estimation of the coefficients of the phase $\phi(t)$ in the context of additive noise. Let z_k be a discrete signal introduced by (21). For the estimation of the polynomial phase coefficients, we first assume that the degree of ϕ is known and equal to N. The proposed algorithm uses the particular version $\mathcal{A}_N(z,\nu,\tau)$ of the GAF and sequentially estimates the coefficients, starting with the highest degree term as follows.

Algorithm 1

1. Initial conditions :
   $m = N$ and $y_k^{(m)} = y_k, \quad 1 \leq k \leq N_e$
2. Choose τ and compute
   $$\widehat{a}_m \triangleq \frac{1}{m!(T_e\tau)^{m-1}} \arg\max_{\nu} |\underline{\mathcal{A}}_m(y,\nu,\tau)| \tag{25}$$
3. Let $y_k^{(m-1)} = y_k^{(m)} e^{-j2\pi(\widehat{a}_m(kT_e)^m)}$, $1 \leq k \leq N_e$, set $m = m-1$
4. If $m \geq 1$, return to 2, else continue to 5
5. Estimate a_0 from $y_k^{(0)}$ by
   $$\widehat{a}_0 = \frac{1}{2\pi}\text{phase}\left[\log\left\{\sum_{k=1}^{N_e} y_k^{(0)}\right\}\right]. \tag{26}$$

3.2 Estimation of instantaneous frequencies in multiple chirp signals

Let us consider a sampled complex signal z_k modelled as :

$$z_k = \sum_{n=1}^{M} A_n \exp[j2\pi(a_{0n} + a_{1n}kT_e + a_{2n}k^2T_e^2)] + w_k \tag{27}$$

where the amplitudes A_n are assumed to be distinct. We propose an algorithm for estimating the number and the instantaneous frequencies of chirps appearing in the signal (27). This algorithm allows us to determine the number M of chirps in the signal z_k and the polynomial phase coefficients, i.e., the a_{mn}'s. The proposed algorithm is an extension of the one given in [5].

Algorithm 2

$M = 0$ and choose a positive real number α.
While $\frac{1}{N_e}\sum_{k=1}^{N_e}|z_k|^2 > \alpha\sigma_w^2$ do
(i) $M = M + 1$
(ii) Estimate a_{mM}, $m = 1, 2$, by

1. $i = 2$ and $z_k^{(i)} = z_k, 1 \leq k \leq N_e$

2. Choose τ and Compute

$$\widehat{a}_{iM} = \frac{1}{i!(\tau T_e)^{(i-1)}} \arg \max_{\nu} |\underline{\mathcal{A}}_i(z, \nu, \tau)|$$

3. $z_k^{(i-1)} = z_k^i \exp[-j2\pi \widehat{a}_{iM}(kT_e)^i]$
   $k = 1, \ldots, N_e$, set $i = i - 1$

4. if $i \geq 1$ then goto 2 else continue to step (iii)

(iii) $z_k = z_k^{(0)} - \frac{1}{N_e} \sum_{k=1}^{N_e} z_k^{(0)}$,
$z_k = z_k \exp\left[j2\pi \widehat{a}_{1M}(kT_e) + \widehat{a}_{2M}(kT_e)^2\right]$
$k = 1, \ldots, N_e$.

4 APPENDIX

Proof of Proposition 5
As $E[w(k)^i] = 0$ and $E[(w(k))^i (w^*(k))^i] = i! \sigma_w^i \; \forall i > 0$, the moment $E\left[\underline{\mathcal{G}}_N(y, \nu, \tau)\right]$ is given by ($c_k \triangleq C_{N-1}^k$) :

$$\begin{aligned}
& \sum_n E\left[\prod_{k=0}^{N-1} \left(y^{\$k}(n - t_k\tau)\right)^{c_k}\right] e^{-j2\pi n\nu} \\
= & \sum_n \prod_{k=0}^{N-1} E\left[\left(y^{\$k}(n - t_k\tau)\right)^{c_k}\right] e^{-j2\pi n\nu} \\
= & \sum_n \prod_{k=0}^{N-1} \sum_{i=0}^{c_k} C_{c_k}^i \left(z^{\$k}(n - t_k\tau)\right)^{c_k - i} \\
& E\left[\left(w^{\$k}(n - t_k\tau)\right)^i\right] e^{-j2\pi n\nu} \\
= & \sum_n \prod_{k=0}^{N-1} \left(z^{\$k}(n - t_k\tau)\right)^{c_k} e^{-j2\pi n\nu} \\
= & \underline{\mathcal{G}}_N(z, \nu, \tau).
\end{aligned}$$

Proof of Proposition 6
The following three-step method allows us to choose the parameter τ ensuring the independency between $y(n - t_k\tau)$ and $y(m - t_l\tau)$.
1. Find the cases where $y(n - t_k\tau) = y(m - t_l\tau)$ with $n \neq m$. This implies that $n - t_k\tau = m - t_l\tau$ with $n \neq m$. According to the hypothesis, the solution $l = k$ gives $m = n$ which is impossible. The remaining solutions give $\tau = \frac{m-n}{l-k}$ with $m \neq n$ and $k \neq l$.
2. In order to ensure that $y(n - t_k\tau)$ is independent of $y(m - t_l\tau)$ for $n \neq m$, the value of τ, assumed to be positive, must satisfy the condition :

$$\tau \neq |\frac{m-n}{l-k}|, \forall m \neq n \quad \text{and} \quad \forall k \neq l \tag{28}$$

3. One may choose τ greater than the maximum of $|\frac{m-n}{l-k}|$ or less than the minimum of $|\frac{m-n}{l-k}|$ or between these two extreme values of $|\frac{m-n}{l-k}|$. We conclude that there are many values of τ ensuring the independency between the variables $y(n - t_k\tau)$ and $y(m - t_l\tau)$. Under this assumption and condition in (28), the moment $E\left[|\underline{\mathcal{G}}_N(y, \nu, \tau)|^2\right]$ is given by ($c_k \triangleq C_{N-1}^k$) :

$$\begin{aligned}
& \sum_n \sum_{m \neq n} E\left[\prod_{k=0}^{N-1} \left(y^{\$k}(n - t_k\tau)\right)^{c_k}\right] \\
& E\left[\prod_{l=0}^{N-1} \left(y^{\$l}(m - t_l\tau)\right)^{C_{N-1}^l}\right]^* e^{-j2\pi(n-m)\nu} + \\
& \sum_n E\left[\prod_{k=0}^{N-1} \left[\left(y^{\$k}(n - t_k\tau)\right)^{c_k}\right] \left[\left(y^{\$k}(n - t_k\tau)\right)^{c_k}\right]^*\right] \\
= & \sum_n \sum_{m \neq n} \left[\prod_{k=0}^{N-1} \left(z^{\$k}(n - t_k\tau)\right)^{c_k}\right] \\
& \left[\prod_{l=0}^{N-1} \left(z^{\$l}(m - t_l\tau)\right)^{C_{N-1}^l}\right]^* e^{-j2\pi(n-m)\nu} + \\
& \sum_n \prod_{k=0}^{N-1} E\left[\left[\left(y^{\$k}(n - t_k\tau)\right)^{c_k}\right] \left[\left(y^{\$k}(n - t_k\tau)\right)^{c_k}\right]^*\right] \\
= & |\underline{\mathcal{G}}_N(z, \nu, \tau)|^2 + A^{2^N}(N_e - (N-1)\tau) K(N, \text{SNR})
\end{aligned}$$

Where

$$K(N, \text{SNR}) = \left[\prod_{k=0}^{N-1} \left[\sum_{i=0}^{c_k} (C_{c_k}^i)^2 i! (\frac{1}{\text{SNR}})^i\right] - 1\right]$$

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