DAMPED SINUSOIDAL SIGNAL RECONSTRUCTION USING HIGHER-ORDER CORRELATIONS

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ABSTRACT

In this paper the reconstruction of deterministic damped sinusoidal signals from a one-dimensional slice of their multiple correlations is analysed. Signal correlations are estimated using a new higher-order correlation estimator, which allows the exponentially damped structure of the signal to be maintained in any horizontal slice of correlations. This characteristic is of utmost importance for the subsequent application of a linear method to estimate the signal parameters and thus reconstruct the signal. Simulations results show that the correlation-based approach gives better reconstructed signals than data-based methods (KT method) when coloured noise contaminates the signal.

1. INTRODUCTION

A very common problem in signal processing is the estimation of damped sinusoidal signal components from a noisy record of data [1]-[5]. In fact, this modelling provides a convenient methodology in the analysis of the response of an electronic system to a burst of electromagnetic energy, the analysis of modes in conducting structures, and in diverse fields such as radio direction finding, high-resolution imaging of moving targets and multiple transient signal processing. In particular, the measured real-valued data y(n) are modelled as:

$$x(n) = \sum_{m=1}^{2M} a_m e^{s_m n} = \sum_{m=1}^{M} 2a_m e^{\sigma_m n}\cos(2\pi f_m n) \qquad n=0...N-1 \tag{1}$$

where the numbers a_m m=1,2...2M are the amplitudes and $s_m = \sigma_m + j2\pi f_m$, m=1,2...2M are the complex frequencies which occur in complex conjugate pairs of equal amplitude, and σ_m are the damping factors and f_m the pole frequencies. The problem addressed here deals with the estimation of the frequencies, damping factors, and amplitudes from a finite number of observed data, possibly contaminated with additive Gaussian noise. This problem becomes very difficult when the number of available data is small and the signal to noise ratio is low. The main difficulty is to estimate the damped frequencies of the signal, since the amplitude estimation reduces to a standard least-squares problem [1]. When the signal is contaminated with coloured noise, a cumulant-based approach provides an appropriate solution to this problem, since higher-order cumulants are blind to additive Gaussian noise, white or coloured [2]. Papadopoulos and Nikias in [3] suggest computing the third-order correlations or fourth-order cumulants of noisy data using a standard estimator for a single data record. However, the deterministic errors associated with the use of the standard biased estimator [4] are considerably higher than stochastic errors due to noise. In [4] is proposed the use of a new estimator which overcomes the problem with the biased estimator and allow some slices in the correlation space to be modelled with the same exponentially damped sinusoidal structure as data.

This paper is concerned with damped sinusoidal signal reconstruction in noise using a one-dimensional (1-D) slice of higher-order correlations. An alternative form of the covariance-type estimator presented in [4] is suggested. This estimator extends the useful range of horizontal slices to allow the exponentially damped structure in a 1-D correlation sequence to be maintained. These results are summarized in a consistent algorithm that collects recent linear systems identification proofs. This algorithm is applied to the reconstruction of real signals and the performance is analysed by Monte Carlo simulations. Simulation results show the effectiveness of the proposed signal reconstruction algorithm in terms of bias and mean-square error of the parameter estimates and coloured noise sensibility.

2. HIGHER-ORDER CORRELATION-BASED APPROACH

Consider the noiseless signal defined in eq.(1). We are concerned with the estimation of the frequencies and damping factors from a 1-D slice (full-rank slice) of the higher-order correlation sequence of the energy signal x(n). In the third-order case, the third-order correlation sequence is defined as:

$$R_x(\tau_1,\tau_2) \doteq \sum_{n=-\infty}^{\infty} x(n)x(n+\tau_1)x(n+\tau_2) \qquad \tau_1,\tau_2=0,\pm1,\pm2 \; ... \tag{2}$$

One dimensional slices of third-order correlations $r_x(\tau)$ can be defined by taking $\tau_2=a\tau+b$ in eq.(2) where *a* and *b* are the slope and intercept respectively of the line along the moments plane.

The signal parameters could be recovered from a 1-D slice in the moments plane provided that the 1-D slice can be modelled as the sum of 2M decaying complex exponentials oscillating with the same frequencies and damping factors as the original signal. Since the 1-D slice behaves like a damped exponential signal, the KT method can be applied, and so the frequencies and damping coefficients of the signal can be identified.

The question that arises here is when the 1-D slice can be modelled as the sum of M damped exponentials. We can distinguish two cases:

2.1 True Higher-order Correlations

If we know the energy signal elsewhere ($\forall$ n $\geq$ 0), we can construct the true third-order correlation sequence without "deterministic" errors [4]. Several cases appear depending on the quadrant in the moment plane the 1-D slice lies on. These cases are collected in the following proposition [4]:

Proposition 1.- Only horizontal slices in the first and fourth quadrants and diagonal slices in the third quadrant allow the original damped exponential structure of data for the moment sequence to be preserved.

2.2 Estimated Higher-order Correlations

In most practical cases, we only have a finite subset of data samples, namely n=0,...,N-1 and we want to estimate *nm* moments from these data. In this case there will be "deterministic" errors associated with imperfect estimation of correlations. These errors can eliminate the fundamental property of higher-order correlations which allows the signal parameters to be recovered, i.e. the moment sequence should be modelled as a sum of exponentially damped sinusoids even for the finite data case. The higher-order correlation of the deterministic signal can be estimated using the standard biased estimator [3]:

$$\hat{R}_x(\tau_1,\tau_2) = \frac{1}{N}\sum_{n=S_1}^{S_2} x(n)x(n+\tau_1)x(n+\tau_2)$$
$$S_1 = \max(0,-\tau_1,-\tau_2)$$
$$S_2 = \min(N-1, N-1-\tau_1, N-1-\tau_2) \qquad (3)$$

Using this estimated moment sequence, only certain horizontal slices (with high lags) in the first quadrant can be used to model $r_x(\tau)$ as a sum of M decaying complex exponentials. In this case the diagonal line in the third quadrant used in [3] is not valid.

In order to avoid the problems associated with a finite number of data in the estimations provided by the standard moments estimator [4], a new type of estimator, called the covariance-type (cov-type) estimator was introduced in [4]. This estimator can be redefined so that every horizontal slice in the moments plane retains signal structure. If $R_x(\tau,\tau)$ denotes the estimated third-order correlation sequence of signal x(n) and τ_{1max} and τ_{2max} are the maximum-computed lags for τ_1 and τ_2, the alternative form for the covariance-type estimator is:

$$\hat{R}_x(\tau_1,\tau_2) = \frac{1}{N}\sum_{n=T_1}^{T_2} x(n)\, x(n+\tau_1)\, x(n+\tau_2)$$
$$\tau_1=0,\pm1,\ldots\tau_{1max} \qquad \tau_2=0,\pm1,\ldots\tau_{2max} \qquad (4)$$
$$T_1 = \max_{\tau_1\tau_2}(0,-\tau_1,-\tau_2) \quad T_2 = \min_{\tau_1\tau_2}(N-1, N-1-\tau_1, N-1-\tau_2)$$

which has been obtained bounding the limits of the summation in eq.(11) for their maximum and minimum values. With this definition of the cov-type estimator, all the horizontal slices of the higher-order correlations are useful, as is stated in the following proposition:

Proposition 2.- If the correlation sequence is estimated according to eq.(4), horizontal slices in all the quadrants retain the same structure of damped exponentials as the data. This allows the KT-method [1] to be used with the estimated moments instead of data to recover signal frequencies.

Proof. Taking into account eq.(1) and eq.(4), the estimated correlation sequence will be:

$$\hat{r}_x(\tau) = R_x(\tau,a\tau+b) = \frac{1}{N}\sum_{ijk}^{2M} \frac{a_i a_j a_k}{1-e^{s_i s_j^* s_k^*}} e^{s_j^*\tau} e^{s_k^*(a\tau+b)} \qquad (5)$$
$$e^{(s_i s_j^* s_k^*)T_1}\left(1-e^{(s_i s_j^* s_k^*)(T_2-T_1+1)}\right) \qquad \tau=0,\pm1,\ldots\pm nm$$

If the slope *a* is chosen equal to 0, and since the limits of the summations T_1 and T_2 are not functions of τ, the estimated correlation sequence maintains the same structure of data, thus avoiding a link among different frequencies as occurs in the standard estimator. Then, when a 1-D slice is estimated in this fashion, every horizontal slice will be possible and this estimator takes the form shown in (6) for each quadrant:

$$\textit{HS quadrant } 1 \quad \frac{1}{N}\sum_{n=0}^{N-1-\tau_{max}} x(n)x(n+\tau)x(n+b)$$
$$\textit{HS quadrant } 2 \quad \frac{1}{N}\sum_{n=\tau_{max}}^{N-1-b} x(n)x(n+\tau)x(n+b)$$
$$\textit{HS quadrant } 3 \quad \frac{1}{N}\sum_{n=\tau_{max}}^{N-1} x(n)x(n+\tau)x(n+b) \qquad (6)$$
$$\textit{HS quadrant } 4 \quad \frac{1}{N}\sum_{n=b}^{N-1-\tau_{max}} x(n)x(n+\tau)x(n+b)$$

In addition, in many practical cases the signal is contaminated with additive noise. Therefore, let us suppose that the observed data sequence consists of N samples from 2M exponentially damped signals in additive Gaussian noise:

$$y(n) = x(n) + w(n) \qquad n=0,1,...N-1 \tag{7}$$

where x(n) is defined in eq.(1) and the additive noise w(n) is a zero-mean, coloured or white stationary Gaussian process, independent of the signal. Assuming a large number of realizations available and defining the signal y'(n) as x'(n)+w(n), x'(n) being the signal in eq.(1) with the mean removed, the third-order cumulant sequence of y'(n) can be expressed [4]:

$$c_{y'}(\tau_1,\tau_2) \doteq E\left\{\frac{1}{N}\sum_{n\,S_1}^{S_2} y(n)\, y(n+\tau_1)\, y(n+\tau_2)\right\} = \frac{1}{N}\sum_{n\,T_1}^{T_2} x(n)\, x(n+\tau_1)\, x(n+\tau_2) = \hat{R}_{x'}(\tau_1,\tau_2) \tag{8}$$

where the summation have been included since the process is not stationary, and it has been taken into account that the signal is independent from the noise and that the third-order moments of a Gaussian process is zero. From eq.(6) it can be seen that the third-order cumulant sequence is the third-order correlation sequence of x'(n), to which can be applied proposition 2 to model it as a damped exponential model with 2M+1 sinusoids (2M frequencies of the original signal plus an additional frequency at zero).

Likewise, the same results can easily be extended to fourth-order cumulants, with both proposition 1 and 2 remaining valid in particular.

2.3 Time-domain signal reconstruction algorithm

Using the estimator proposed in eq.(4), the algorithm of time-domain signal reconstruction can be summarized as follows:

1) Estimate the third/fourth-order correlations of signal x(n) along a prescribed horizontal slice using the cov-type estimator.
2) Use the identifiability results of [4] to obtain a rank-2M matrix with the estimated correlation sequence.
3) Compute the 2M-truncated least-squares solution in terms of an SVD according to the KT method [1].
4) Evaluate the amplitudes by solving a Vandermonde system with the measured data in the least-squares sense.

3. SIMULATION RESULTS

In this section we evaluate the performance of the time-domain reconstruction signal approach when few data are available, estimating the fourth-order signal correlations using the cov-type estimator defined in eq.(4) (called the COV-FOC method). In the simulations, we use fourth-order correlations or cumulants since they show a clear superiority compared to third-order moments. For purposes of comparison, the Kumaresan-Tufts (KT) method is also employed to reconstruct the signal. In the simulations, the signal consisted of two unity amplitude real damped sinusoids given by:

$$y(n) = e^{\sigma_1 nT}\cos(2\pi f_1 nT) + e^{\sigma_2 nT}\cos(2\pi f_2 nT) + w(nT) \qquad n = 0...N-1 \tag{9}$$

where σ_1=-0.2, f_1=0.42, σ_2=-0.1, f_2=0.52, and T=0.5. The noise as coloured Gaussian noise generated by passing a white Gaussian process through a FIR filter with an impulse response given [3] by h=[0.5, 0.6, 0.7, 0.8, 0.7, 0.6, 0.5, 0, 0, 0.5, 0.6, 0.7, 0.8, 0.7, 0.6, 0.5]. The signal-to-noise ratio is defined as SNR= 10 log($1/\sigma^2$). In the experiments, the cumulant sequence consisted of 25 cumulants estimated from 30 data, and cumulant and data matrices were constructed with a selected filter order of 13. The frequencies and damping factors were computed for each realization and then the amplitudes were evaluated by least-squares fit to the data for 500 independent runs according to section 2.3. To demonstrate the SNR improvement achieved by the COV-FOC method, Figure 1 plots the original transient and the reconstructed signal obtained from the COV-FOC and KT methods when the SNR was set up to 10 dB. Notice that the signal estimate from the COV-FOC method shows good agreement, but the KT estimate is influenced by imperfect estimation of the frequency of the smaller damping coefficient due to the coloured noise.

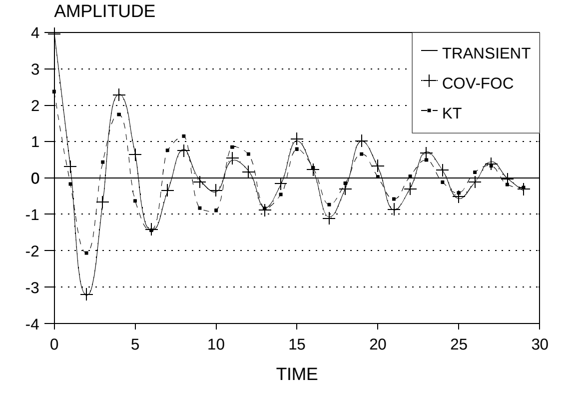

Figure 1. True transient signal and reconstructed signal using KT and COV-FOC methods at SNR=10 dB colored noise.

To show the independence of the COV-FOC method from the autocorrelation sequence of the coloured noise, we vary the FIR system used to filter the noise, generating new coloured Gaussian noise sequences by passing white Gaussian noise through a second-order FIR filter whose zeros are located at different frequencies. The results for the least damped frequency with the COV-FOC and KT methods are shown in Figure 2. In this figure, frequency zeros of the FIR filter were lying in the interval [0.1 0.9]. Notice that the COV-FOC method estimates this frequency independently from the noise spectrum, whereas the KT method only estimates this frequency when the zeros of the FIR filter lie on the signal frequency interval, i.e. the noise spectrum lacks frequency components near the signal frequencies.

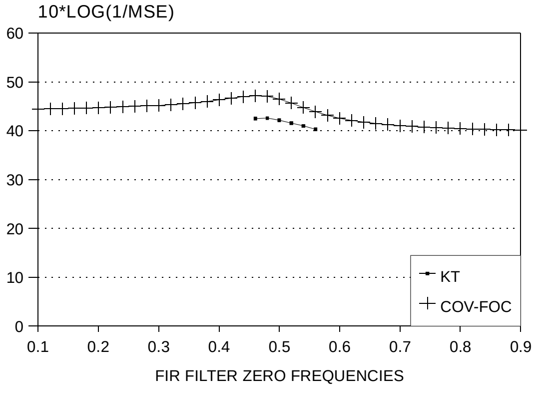

Figure 2. Influence of colored noise spectrum in frequency estimates using COV-FOC and KT methods.

Finally, a test to compare the capacity of the proposed COV-FOC method to recover signal frequencies from real data with respect to the method [3] based on the use of the standard estimator of correlations (BIA-FOC method) was carried out. The signal consisted of a real damped sinusoid oscillating at frequency 0.52 with damping factor -0.1. The noise is coloured Gaussian noise generated as above. The frequency and damping factor were estimated from 25 cumulants estimated from 30 data using the COV-FOC and BIA-FOC methods. The filter order was chosen as 13 and T=0.5. Figures 3(a) and (b) shows the zeros of the prediction-error filter polynomial for 50 independent trials in coloured noise using COV-FOC method at SNR set up to 15 dB and 5 dB respectively, and Figures 3(c) and 3(d) for the BIA-FOC method. These plots clearly demonstrate that the COV-FOC method performs better, accurately estimating the real frequency. On the other hand, the BIA-FOC method fails to recover signal frequency. This is due to the fact that the BIA-FOC method is highly affected by "deterministic errors" appearing several sinusoids oscillating at frequencies near to zero. This situation results in an estimated cumulant sequence strongly perturbated by the imperfect modelling. This behaviour is made clear in Figure 3(c) and (d), where multiple zeros appear near the real axis and a persistent bias alters the estimates.

4. CONCLUSIONS

Higher-order correlations can be successfully applied to the reconstruction of deterministic transients (damped sinusoidal signals) contaminated by Gaussian noise. To accomplish this, an estimator which allows any horizontal slice of higher-order correlations to be modelled with the same frequencies as the original data is proposed. Simulation examples confirm the expected theoretical advantage of the proposed signal reconstruction algorithm compared to data-based methods when the number of available data is small.

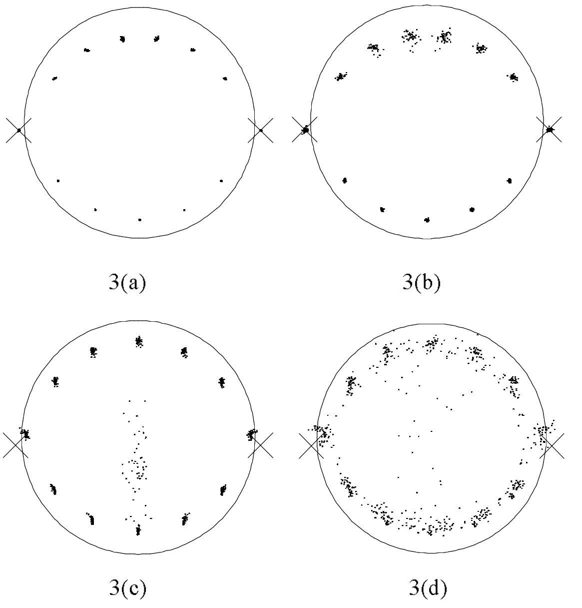

Figure 3. Zeros of the prediction error filter polynomial in 50 independent trials (the intersecting radial lines and arcs show the true location of the zeros)

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