USE OF FOURTH-ORDER STATISTICS FOR NON-GAUSSIAN NOISE MODELLING: THE GENERALIZED GAUSSIAN PDF IN TERMS OF KURTOSIS

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ABSTRACT

In this paper non-Gaussian noise modelling is addressed. HOS-based parametric pdf models are investigated in order to provide realistic noise modelling by means of easy and quick estimation of needed parameters.

Attention is focused on the generalized Gaussian pdf. This model, generally depending on a real theoretical parameter c, difficult to estimate from data, is proposed expressed in terms of the fourth-order parameter kurtosis β_2 by introducing the analytical relationship between c and β_2. The model is compared with well-known pdfs and used in the design of a LOD test.

1 INTRODUCTION

This work is addressed to *realistic characterization of generic background noise* aimed at the optimization of *signal detection* in *non-Gaussian environments*.

Detection is dealt with as *binary hypothesis testing* in the context of *statistical inference* [1]: the decision between the two hypotheses of the presence (H_1) or the absence (H_0) of a transmitted signal $\{s_i,\ i=1, .., M\}$ is made on the basis of acquired observations $\{y_i,\ i=1, .., M\}$ [1]; the noise, $\{n_i,\ i=1, .., M\}$, is assumed additive, independent and identically distributed (iid), stationary, unimodal, generally non-Gaussian.

Among the main targets addressed, *easy applicability to real cases* is focused, in terms of realistic noise modelling, easy and realistic estimation of model parameters, and robustness to variable boundary conditions.

Symmetric probability density function (pdf) models are considered. In order to satisfy the mentioned requirements of easy applicability of a model to real cases, the investigation is addressed to express generalized noise pdfs, usually depending on parameters difficult to be estimated from real data samples, in terms of *Higher-Order-Statistics* (HOS) parameters, which are very easy and quick to be extracted from data and are particularly suitable for quantifying deviation from Gaussianity [2].

As conventional signal processing algorithms based on the Second Order Statistics, optimized in presence of Gaussian noise, may decay in non-Gaussian noise, various works used HOS theory [2] as signal-processing basis for noise analysis and detection optimization; however, some methods work only with non-Gaussian signals [3][4][5] or only in Gaussian noise [5][6][7]; some are not optimized for low SNR values [3].

In this paper, attention is focused on the *generalized Gaussian* function; it depends on a real parameter, c, which is not easy to estimate from data. Nevertheless, c presents a physical meaning, as linked with the pdf *sharpness*. The HOS parameter which better describes sharpness variability is the fourth-order *kurtosis*, β_2. The analytical relationship between c and β_2 and the range of kurtosis in which the resulting pdf model can be applied are introduced. The resulting symmetric function has the same characteristics of the generalized Gaussian, and is a realistic noise-pdf model for $1.865<\beta_2\leq 30$ (hence for both sub- and super-Gaussian pdfs). It is compared with another kurtosis-based generalized symmetric function, the Champernowne model [8][9], which results less general.

In order to detect signals in the critical case of low SNR values (in the range [-20, 0] dB), the statistical testing approach selected is a *Locally Optimum Detector* (LOD) [1]. The new pdf proposed is applied in the design of a LOD test, used for detecting constant weak signals corrupted by real underwater acoustic noise [10][11].

2 DESCRIPTION OF THE KURTOSIS-BASED MODEL AND ITS APPLICATION TO THE LOD TEST

In the context of noise modelling, one of the most noticeable ways in which estimated noise distributions deviate from Gaussianity is in kurtosis β_2, i.e., the ratio of the fourth and the square of the second moments. It is equal to 3 in the Gaussian case; the sharpness of the pdf shape is higher (lower) than the corresponding Gaussian function when β_2 is larger (smaller) than 3. A good model for generalized symmetric pdfs has *variable sharpness*.

One of the well-known symmetric pdf models with variable sharpness is the generalized Gaussian, which depends on the real parameter c:

$$p_{gg}(x) = \frac{\gamma c}{2\Gamma(1/c)} e^{-|\gamma(x-\mu)|^c} \quad (1)$$

where $\{x\}$ is generic iid noise with mean value μ and variance σ^2.

The model variability is due to the two parameters γ and c.

γ influences the deviation of samples from the mean value, since it is expressed in terms of the variance:

$$\gamma = \sqrt{\frac{\Gamma(3/c)}{\sigma^2 \Gamma(1/c)}}, \quad (2)$$

where $\Gamma(.)$ is the standard Gamma function:

$$\Gamma(k) = \int_0^{+\infty} y^{k-1} e^{-y} dy \quad (3)$$

c is a theoretical parameter that influences the model sharpness, but cannot be directly estimated from data samples; hence the relationship between c and β_2 is introduced. It derives from the β_2 definition in terms of the pdf and is expressed by the following formula:

$$\beta_2^x = \frac{m_4^x}{(m_2^x)^2} = \frac{E\{(x-\mu)^4\}}{\left(E\{(x-\mu)^2\}\right)^2} = \frac{\Gamma(5/c)\Gamma(1/c)}{(\Gamma(3/c))^2} \quad (4)$$

Because of the $\Gamma(.)$ function definition, it is impossible to express c in terms of β_2 with an analytically exact expression. Hence an approximation is required. A good approximation was found by applying the Least Squared Method (LSM) on a generic second-order, monotonic analytical expression of $\beta_2 \approx \beta_2(c)$:

$$\beta_2 \approx \frac{1.865 c^2 + \alpha_1 c + \alpha_2}{c^2 + \delta_1 c + \delta_2} \quad (5)$$

With this parametric expression the same asymptotic behaviour of (4) was approximately maintained:

$\beta_2 \approx 3$ for c=2 (Gaussian case); $\beta_2 \approx 6$ for c=1 (Laplace case).

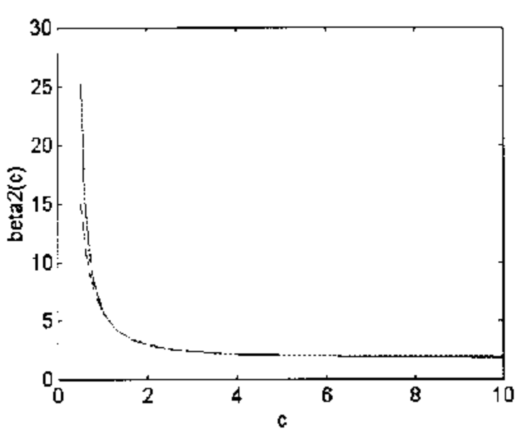

Fig. 1. Comparison between the true (solid line) and the approximated (dashed line) function $\beta_2 = \beta_2(c)$.

The LSM results follows:

$$\beta_2 \approx \frac{1.865(c+0.12)^2}{(c+0.12)^2} \quad (6)$$

A comparison between the shape of the exact function (4), numerically computed, and the proposed approximation is shown in Fig. 1.

By inverting the monotonic function (6), the following expression can be found:

$$c \approx \sqrt{\frac{5}{\beta_2 - 1.865}} - 0.12 \quad (7)$$

which is defined for $\beta_2 > 1.865$ and can be considered a good approximation of the exact expression for $\beta_2 \leq 30$. This range includes about all the kurtosis values which can be measured in real applications [8].

This formula allows one to express $p_{gg}(x)$ in terms of β_2. Its validity is confirmed by observing that for $\beta_2 > 3$ the resulting pdf has *heavy tails*, as expected [11]. Figure 2 shows a family of generalized Gaussian functions as β_2 varies.

Another useful kurtosis-based symmetric pdf with variable shape is the Champernowne function [8]; however it is less general than the aforesaid model, as it can be applied only if the β_2 value falls in the range $1.8 \leq \beta_2 \leq 4.2$ (hence the Laplace case is not included), and is realistic only if noise components have a hyperbolic distribution of power [8].

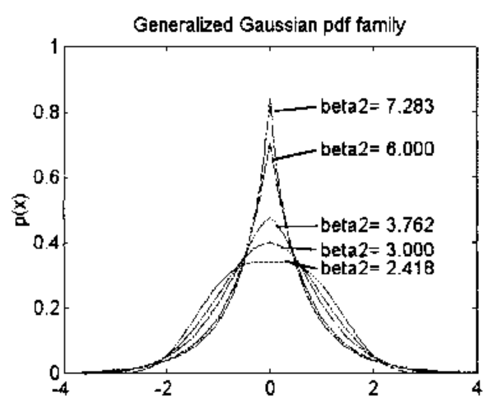

Fig. 2. Generalized Gaussian family (μ=0; σ^2=1).

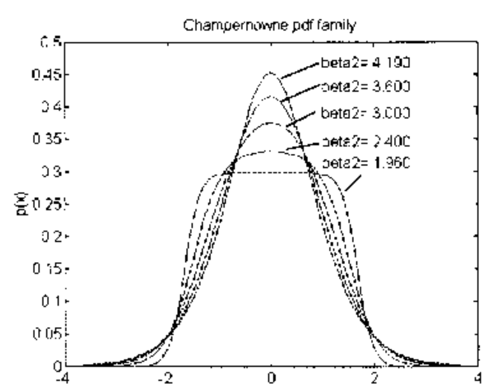

Fig. 3. Champernowne pdf family (μ=0; σ^2=1).

Figure 3 shows a number of Champernowne functions as β_2 varies.

In both cases the additional information introduced in the models for a more realistic and generalized noise statistical characterization is contained in a single 4th-order parameter (β_2), being very simple and quick to estimate.

Another class of parametric distribution functions useful for describing certain kinds of noise processes consists of the α-stable distributions ($0<\alpha<2$) [12], deriving form the Fractional Lower-Order Statistics; in [12] the *symmetric α-stable functions* are investigated and applied. Their main limitation is that they can describe in a realistic way only *impulsive* noise; for this reason they are not taken into account in this work, devoted to *generalized* noise characterization.

3 APPLICATION TO THE DESIGN OF A LOD TEST

The kurtosis-based generalized Gaussian and Champernowne models are suitable for the design of LOD tests [1], as in both cases the non-linearity $g_{lo}(x)$ and the maximum asymptotic relative efficiency ρ can be expressed analytically in terms of elementary functions.

In this paper attention is focused on the problem of detecting constant M-sample-long time signals having amplitude θ when they reach the receiver ($\theta \to 0$).

The LOD test criterion for constant signals [1] is based on the following expression linking the test statistics λ_{lo} and the statistical threshold T_α, given the significance level α:

$$\lambda_{lo} = \sum_{i=1}^{M} g_{lo}(y_i) \begin{cases} \geq T_\alpha & => H_1 \text{ is decided} \\ < T_\alpha & => H_0 \text{ is decided} \end{cases} \tag{8}$$

where:

$$g_{lo}(y) = -\frac{\dot{p}(y)}{p(y)} \tag{9}$$

$$P_{FA} = \alpha = \int_{T_\alpha}^{+\infty} p_{\Lambda_{lo}/H_0}(\lambda_{lo} / H_0)\, d\lambda \tag{10}$$

$p(x)$ is the noise pdf model, $\{y_i,\ i=1,\ldots,M\}$ is the received observation sequence on the basis of which to decide.

Further than the non-linearity function, one of the most significant parameter for determining the detector performances from a theoretical point of view is the maximum Asymptotic Relative Efficiency (ARE) ρ [1], defined in terms of the noise variance and pdf model as follows:

$$\rho = \sigma^2 \int_{-\infty}^{+\infty} [\dot{p}(x)]^2 [p(x)]^{-1}\, dx \tag{11}$$

It is to notice that no constraint is requested about the statistics of the signal to detect; moreover the test needs only the value of P_{FA}, α, fixed by the user.

On the other hand, the main limitations of a LOD approach consist in the following characteristics:

1. it needs complete a-priori knowledge about the signal when it is acquired (in terms of time shape); this aspect is particularly critical if distortion phenomena occurs during the propagation: in these cases, not only complete a-priori knowledge on the transmitted signal, but also a realistic channel model have to be available;
2. only for few pdf models the test threshold T_α can be computed analytically on the basis of the λ_{lo} expression (8); otherwise, the threshold has to be computed by means of numerical or empirical procedures.

In the generalized Gaussian case the following expressions for $g_{lo}(.)$ and ρ can be found:

$$g_{lo,gg}(y) = c\gamma^c\, |y-\mu|^{c-1}\, sgn(y-\mu) \tag{12}$$

$$\rho_{gg} = \frac{c^2 \Gamma(2-1/c)\Gamma(3/c)}{\Gamma(1/c)} \tag{13}$$

(notice that $\beta_2<25.2$ is needed for having a finite ρ).

The respective graphs are presented in Figs. 4 and 5. For analytical details on the LOD test based on the Champernowne noise model, see [8]. These non-linearity and maximum ARE graphs can be compared with similar graphs given in [11] for other non-Gaussian known pdfs.

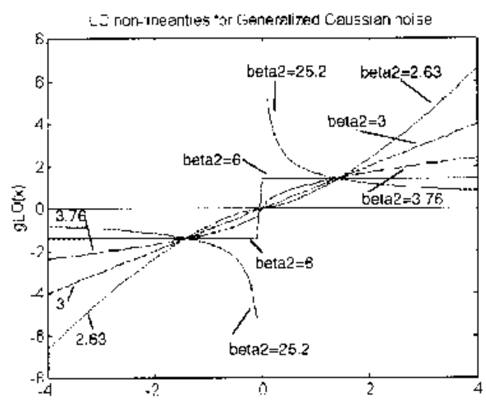

Fig. 4. Graphs of $g_{lo}(x)$ as β_2 varies ($\sigma^2=1$).

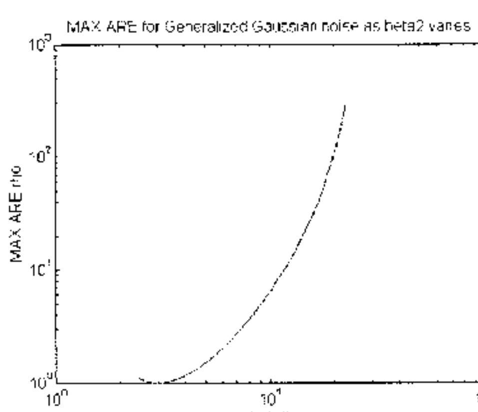

Fig. 5. Graphs of ρ in terms of β_2 varies ($\sigma^2=1$).

4 EXPERIMENTAL RESULTS

The two LOD tests, presented in Section 2 and based on the generalized Gaussian and the Champernowne pdfs respectively, were applied for the detection of known deterministic signals corrupted by real underwater acoustic ship-traffic-radiated noise. Selected noise sequences, sampled at a sample frequency F_s=2000 Hz, were analyzed and characterized at average by the estimated parameters μ=0, σ=1650, β_2=2.45, as described in detail in [9][10][11].

LOD performances are presented by means of experimental curves of the detection probability as SNR varies, given a certain value of the Probability of False Alarm, P_{FA} (P_{FA}=5%) and a fixed number of samples M (M=1000). Results are shown in Figs. 6 and 7.

As expected, detection performances improve as much as the pdf models are generalized and depend on a large number of parameters.

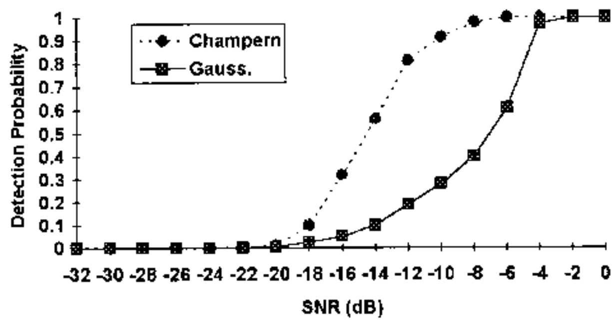

Fig. 6. Comparison of LOD tests based on the Gaussian and Champernowne pdfs.

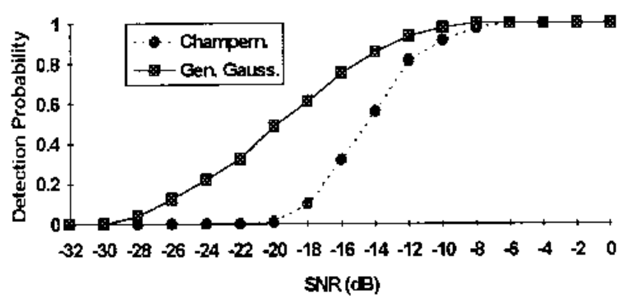

Fig. 7. Comparison of LOD tests based on the Champernowne and the generalized Gaussian pdfs.

5 CONCLUSIONS

The well-known *generalized Gaussian pdf* has been focused as a very useful analytical parametric generalized model for detection optimization in real environments. It is introduced as depending on the fourth-order parameter *kurtosis*, instead of on a theoretical parameter in order to make easier and optimize its application to the statistical characterization of real stochastic data. This model has been applied in the design of a LOD test and compared with another kurtosis-based pdf model, the Champernowne pdf, which has resulted less general.

The main limitation of the kurtosis-based generalized Gaussian function is its *symmetry*: in order to include also asymmetric shapes, a third-order parameter, e.g., the skewness, should be inserted as additional pdf parameter.

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