Application of the Structured Total Least Norm technique in spectral estimation

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ABSTRACT

In the problem of estimating parameters of exponentially damped sinusoids, an improved variant of Kung's method, called HTLS and based on the use of a Hankel data matrix, the singular value decomposition and the total least squares technique, has been proposed and proven to be accurate and efficient. In this paper, a more accurate estimator HTLN is proposed. It starts from the same Hankel data matrix, but uses a new technique, called structured total least norm, prior to the HTLS estimator. This technique computes the solution of a *structured* overdetermined linear system, $\mathbf{AX} \approx \mathbf{B}$, with possible errors in both $\mathbf{A}$ and $\mathbf{B}$. The better accuracy of the STLN and HTLN techniques is shown by means of computer simulations.

1 INTRODUCTION

This paper addresses the problem of parameter estimation of a sum of K exponentially damped sinusoids embedded in noise, which is a kernel problem in many signal processing applications such as Nuclear Magnetic Resonance spectroscopy, direction-of-arrival estimation, radar, sonar, etc. The following data model is taken:

$$x_n = \sum_{k=1}^{K} c_k e^{(-d_k + j2\pi f_k)n\Delta t} + e_n,\ n = 0, 1, \ldots, N-1$$

where $j = \sqrt{-1}$, c_k is the complex-valued linear parameter; d_k and f_k are respectively the damping factor and frequency of the kth peak; Δt is the sampling interval and e_n is complex white Gaussian noise. $z_k = \exp[-d_k \Delta t + j(2\pi f_k \Delta t)]$ is called the kth pole of the signal. The pole estimator used in this paper is HTLS, which is an improved variant of Kung's method [1] and based on the use of a Hankel data matrix, the SVD, and total least squares (TLS). It is a subspace-based method, computing from the Hankel data matrix a "signal" and a "noise" subspace and extracting the pole information from the "signal" subspace of rank K. The computed matrix representing the "signal" subspace, however, does not preserve the Hankel structure anymore and hence needs improvement.

Recently, the structured total least norm (STLN) [3][4] technique has been proposed for computing the solution of a *structured* over-determined linear system, $\mathbf{AX} \approx \mathbf{B}$, with possible errors in both $\mathbf{A}$ and $\mathbf{B}$. STLN differs from the classical TLS technique in that it preserves the special structure of $\mathbf{A}$ or $[\mathbf{A}\ \mathbf{B}]$, such as Hankel, Toeplitz or sparse, and minimizes a measure of error in the discrete L_p norm, where $p = 1, 2$ or ∞. This technique is applied in the next section to find the best rank-K structure-preserving approximation, $[\mathbf{A}+\mathbf{E}\ \ \mathbf{B}+\mathbf{F}]$, of the original Hankel data matrix, $[\mathbf{A}\ \mathbf{B}]$, as well as the solution, $\mathbf{X}$, from which the noise subspace and then the signal subspace can be found for pole extraction. The new pole-estimation technique is called HTLN. It first uses the STLN approximation and then the pole-extraction technique of HTLS.

The STLN approximation used in HTLN is iterative. It filters out the noise from the measured signal. In this sense, STLN can be used as a signal enhancement method, similar to some SVD-based enhancement algorithms such as Cadzow's (CA) method [5], the minimum variance (MV) method [6] and the structured total least squares (STLS) [7] technique. All these methods find a rank-K approximation $[\mathbf{A}+\mathbf{E}\ \ \mathbf{B}+\mathbf{F}]$ of the original Hankel data matrix $[\mathbf{A}\ \ \mathbf{B}]$, in different ways, while preserving the matrix structure. In section 3, the STLN technique is compared with the CA, MV and STLS method in signal enhancement, and HTLN is compared with HTLS, as well as CA-HTLS, MV-HTLS, and STLS-HTLS, which are the abbreviated representation of the applied signal enhancement procedures and the parameter estimator used thereafter.

2 STLN AND HTLN

Arrange the modeled data samples into a Hankel matrix:

$$\mathbf{H} = \begin{bmatrix} x_0 & x_1 & \cdots & x_{M-1} \\ x_1 & x_2 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ x_{L-1} & \cdot & \cdots & x_{N-1} \end{bmatrix} = [\mathbf{A}_{L\times K}\ \mathbf{B}_{L\times(M-K)}] \quad (1)$$

where $L > K, M > K, N = L + M - 1$. Since $[\mathbf{A}\ \ \mathbf{B}]$ has rank K when no noise is added, there exists a $K \times (M-K)$ matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$. If noise is added, the corresponding set of linear equations $\mathbf{AX} \approx \mathbf{B}$ is no longer compatible, since $[\mathbf{A} \;\; \mathbf{B}]$ is typically of full rank. The rank-K property of $[\mathbf{A} \;\; \mathbf{B}]$ can be restored by computing appropriate corrections $[\mathbf{E} \;\; \mathbf{F}]$ such that $[\mathbf{A}+\mathbf{E} \;\; \mathbf{B}+\mathbf{F}]$ has rank K, i.e., $(\mathbf{A}+\mathbf{E})\mathbf{X} = \mathbf{B}+\mathbf{F}$ is exactly solvable.

The STLN technique for solving $\mathbf{AX} \approx \mathbf{B}$, where the matrix $[\mathbf{A}_{L\times K} \;\; \mathbf{B}_{L\times(M-K)}]$ is structured, such as Hankel defined in (1), is formulated below:

$$\min_{\mathbf{E},\mathbf{F}} \|[\mathbf{E} \;\; \mathbf{F}]\|_p \qquad p = 1, 2, \infty$$

$$\text{s. t. } \begin{cases} (\mathbf{A}+\mathbf{E})\mathbf{X} = (\mathbf{B}+\mathbf{F}) \\ [\mathbf{E} \;\; \mathbf{F}] \text{ preserves the structure of } [\mathbf{A} \;\; \mathbf{B}] \end{cases}$$

where $\mathbf{E}$ and $\mathbf{F}$ are correction matrices with the same special structure as $\mathbf{A}$ and $\mathbf{B}$. For $p = 2$, STLN reduces to TLS if all elements of $[\mathbf{A} \; \mathbf{B}]$ are subject to error and the structure is not taken into account. The theory and implementation of the STLN algorithm has been treated extensively in [3][4]. For our spectral estimation problem, we apply the STLN algorithm to Hankel structures using the L_2 norm ($p = 2$).

The result of applying the STLN technique to our noise-corrupted data matrix $[\mathbf{A} \; \mathbf{B}]$ in (1) is a rank-K Hankel matrix approximation $[\mathbf{A}+\mathbf{E} \;\; \mathbf{B}+\mathbf{F}]$ and a solution $\mathbf{X}$. From $[\mathbf{A}+\mathbf{E} \;\; \mathbf{B}+\mathbf{F}]$, a "cleaned-up" signal is obtained at convergence, given by the first column and last row of the matrix $[\mathbf{A}+\mathbf{E} \;\; \mathbf{B}+\mathbf{F}]$. Obviously, such a "cleaned-up" (though may be deviated) signal can be written as an exact sum of K exponentially damped sinusoids. This enables several ways of simplification in the subsequent parameter estimator when "cleaned-up" data are used.

When the STLN enhancement is fully convergent, one way of increasing efficiency is to reduce the number of data samples used in the pole-estimation algorithm to as small as $2K-1$ without affecting the resulting estimates. In particular, we can simply arrange the $2K-1$ "cleaned-up" data into a square Hankel matrix $\mathbf{H}$ of size $K \times K$, and compute nonlinear parameters from eigenvalues of the solution $\mathbf{Z}$ in $\mathbf{H}^{\uparrow} = \mathbf{H}_{\downarrow}\mathbf{Z}$ (the up (down) arrow stands for deleting the top (bottom) row), without performing the SVD.

However, the enhancement is typically *not fully* convergent due to the use of higher tolerances for reason of efficiency, and then HTLS, is suggested for estimating the parameters of the enhanced data. Such an STLN-HTLS estimator, can be simplified and improved in efficiency by using the computed STLN solution $\mathbf{X}$. This implies that we do not need to run the HTLS algorithm from the beginning after STLN enhancement. In this way, STLN-HTLS reduces to HTLN (stands for Hankel Total Least Norm), as outlined below.

Outline of the HTLN algorithm

Given. A model-order estimate K, and N data samples $x_n, n = 0, 1, \ldots, N-1$ arranged in the Hankel matrix $[\mathbf{A}_{L\times K} \;\; \mathbf{B}_{L\times(M-K)}]$ as in (1).

Step 1: Signal enhancement. Apply STLN to $[\mathbf{A} \;\; \mathbf{B}]$ such that $[\mathbf{A}+\mathbf{E} \;\; \mathbf{B}+\mathbf{F}]\begin{bmatrix}\mathbf{X} \\ -\mathbf{I}\end{bmatrix} = \mathbf{0}$

Step 2: Computation of signal subspace basis. Compute a QR decomposition of

$$\begin{bmatrix}\mathbf{X} \\ -\mathbf{I}\end{bmatrix} = [\mathbf{Q}_n \;\; \mathbf{Q}_s]\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}$$

where $[\mathbf{Q}_n \;\; \mathbf{Q}_s]$ is unitary, and $\mathbf{R}$ is upper triangular. $\mathbf{Q}_n$ and $\mathbf{Q}_s$ span the null space and signal space of $[\mathbf{A} \; \mathbf{B}]$, respectively.

Step 3: Parameter estimation. Compute the TLS solution $\hat{\mathbf{Z}}$ of the equation $\mathbf{Q}_s^{\uparrow} = \mathbf{Q}_{s\downarrow}\hat{\mathbf{Z}}$, and retrieve the eigenvalues z_k of $\hat{\mathbf{Z}}^H$, where ^{H} denotes Hermitian conjugate. The f_k, d_k are obtained straightforwardly from z_k.

The improved efficiency of HTLN, compared to STLN-HTLS, stems from the use of the QR decomposition of $\begin{bmatrix}\mathbf{X} \\ -\mathbf{I}\end{bmatrix}$, instead of the SVD of $[\mathbf{A}+\mathbf{E} \;\; \mathbf{B}+\mathbf{F}]$, for finding the orthonormal signal subspace basis.

3 SIMULATION RESULTS

3.1 Simulation Signal And Procedures

This is a complex signal, frequently used in the literature (see, e.g. [6]), with two exponentially damped sinusoids:

$$x_n = e^{-0.01n + j(2\pi 0.2n)} + e^{-0.02n + j(2\pi 0.22n)} + e_n$$

$N = 25$ data samples are taken for the following study. The white Gaussian noise has standard deviation σ_ν on real and imaginary components, respectively. The signal-to-noise ratio (SNR) is defined as $10\lg(1/(2\sigma_\nu^2))$.

The maximum number of iterations of the iterative enhancement methods is set to 100. The tolerance ϵ [9] in the stop criterion of iterative methods is $\epsilon = 10^{-10}$. Different techniques such as CA, MV, STLS, and STLN, which solve the incompatible set $\mathbf{AX} \approx \mathbf{B}$ by adding different Hankel correction matrices $[\mathbf{E} \;\; \mathbf{F}]$ to the noisy, full rank Hankel matrix $[\mathbf{A} \;\; \mathbf{B}]$, are compared in $\|[\mathbf{E} \;\; \mathbf{F}]\|_F$ averaged over 200 noise realizations. The performance of different enhanced estimation methods CA-HTLS, MV-HTLS, STLS-HTLS, and HTLN are studied, in terms of respectively resolution and parameter accuracy, by comparing the number of failures and the root mean-squared error (RMSE). The RMSE values of the estimates of the signal parameters f_k, d_k, and c_k are computed from the good runs out of 200 noise realizations, excluding failures and non-convergent cases. A failure (among the convergent runs by the iterative methods,) occurs if not all peaks are resolved within specified intervals, 0.2 ± 0.0094 Hz and 0.22 ± 0.0106 Hz in this example. The size $L \times M$ of the Hankel data matrix used in the HTLS estimator is 13×13 since a square matrix produces the best results [2]. The size $L \times M$ used in the enhancement procedure is 23×3 for MV, STLS and STLN and 13×13 for Cadzow's method, which takes the same dimension as used by the applied estimator HTLS.

3.2 Simulation Results

Table 1 compares the corrections applied to the data matrices or data vectors by the enhancement methods studied, as well as those of other techniques such as LS and TLS that solves $\mathbf{AX} \approx \mathbf{B}$. In all 200 runs, the MV method did not converge within 100 iterations. All other iterative methods converge within 100 iterations. corr_M is the expected Frobenius norm of the correction matrix $[\mathbf{E} \;\; \mathbf{F}]$. If $[\mathbf{E} \;\; \mathbf{F}]$ is Hankel then corr_v represents the expected Frobenius norm of the corresponding correction vector given by the first column and the last row of $[\mathbf{E} \;\; \mathbf{F}]$. Note that the corrections of the TLS method are always minimal, which can be easily proven. The corrections applied by STLN and STLS are about the same, larger than the correction applied by TLS which does not take the Hankel structure into account but lower than the true correction (or simulated noise) and the LS correction. Although MV has even lower corrections after 100 iterations, it does not successfully reduce the rank of the Hankel matrix $[\mathbf{A} \;\; \mathbf{B}]$ to $K = 2$ since it fails to converge within 100 iterations. Hence the correction applied by STLN and STLS is the lowest among the methods that preserve the Hankel structure and reduce the rank to K.

All enhancement methods give very smooth signals when convergence occurs. However, when parameter estimation is performed after signal enhancement, the MV and Cadzow's algorithms need not be used up to convergence when followed by an SVD-based estimator, such as HTLS, since the first iteration is the most decisive one [8]. Therefore, the MV and Cadzow's methods perform only one iteration when used to enhance the parameter estimates in the following study.

Table 1. Expected (averaged over 200 runs) Frobenius norm, corr_M and corr_v, of correction matrices and correction vectors at SNR= 25dB

method	M	corr_M	corr_v
TRUE[a]	3	$4.627e-01$	$2.781e-01$
Cadzow	13	$6.693e-01$	$2.628e-01$
MV	3	$4.087e-01$	$2.476e-01$
STLS	3	$4.272e-01$	$2.581e-01$
STLN	3	$4.272e-01$	$2.581e-01$
LS	3	$4.297e-01$	
TLS	3	$2.650e-01$	
TRUE[a]	4	$5.218e-01$	$2.792e-01$
MV	4	$4.758e-01$	$2.563e-01$
STLN	4	$4.815e-01$	$2.593e-01$
LS	4	$6.837e-01$	
TLS	4	$3.651e-01$	

[a] True correction or simulated noise.

Table 2 lists the results obtained with the different methods. At SNR's as high as 25 dB, no failure in convergence or failure in resolving the two peaks has been observed. HTLN and STLS-HTLS produce almost the same results. These methods obtain lower RMSE values and hence are more accurate than HTLS, CA-HTLS and MV-HTLS. At SNR=20 dB, the iterative methods HTLN and STLS-HTLS still converge well, typically within 30 iterations. But there are some failures for all the methods. The iterative methods have fewer failures than the non-iterative methods HTLS, CA-HTLS and MV-HTLS, and yield lower RMSE. When the SNR is lowered to 15 dB, there are several runs in which HTLN and STLS-HTLS fail to converge within 100 iterations. Among the runs in which convergence occurred, there are still failures. At this SNR, MV-HTLS can better resolve the two interfering peaks since it produces the least number of failures. The RMSE of MV-HTLS and the iterative methods are comparable, all of which are lower than HTLS and CA-HTLS. Looking at the bias and standard deviation, we can see that HTLN and STLS-HTLS reduce the bias for d_1 with a factor 2 to 4 compared to HTLS at all the SNR's. In addition, HTLN and STLS-HTLS yield lower standard deviation for both f_1 and d_1 at almost all the SNR's.

Although HTLN and STLS-HTLS are more accurate than other methods, they are less efficient as can be seen from the number of flops in Table 3. Therefore, one must trade off between the efficiency and accuracy requirement when choosing among iterative and non-iterative methods. The computational cost of these methods gets larger as the number of data samples N is larger or SNR is lower. As well, the memory requirement gets larger when N is increased. More efficient STLN algorithms using line-search techniques are under study.

Table 3. Averaged number of flops using Matlab

methods	SNR 25dB	SNR 20dB	SNR 15dB
HTLS	1.19e+05	1.21e+05	1.23e+05
CA-HTLS	2.23e+05	2.25e+05	2.27e+05
MV-HTLS	1.24e+05	1.24e+05	1.25e+05
STLS-HTLS	8.43e+06	1.38e+07	2.60e+07
HTLN	4.86e+07	6.55e+07	1.15e+08

4 CONCLUSION

A recently proposed structure-preserving total least norm technique, STLN, has been applied to signal enhancement and parameter estimation of exponentially damped sinusoids. An efficient version of the parameter estimator HTLS enhanced by STLN is described and is called HTLN. This technique removes corruption noise effectively and improves resolution and parameter accuracy, especially, the standard deviations of the parameters, when the number of data samples is small.

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Table 2. RMSE, bias and standard deviation of f_1 and d_1

SNR	method	non-convergences	failures	RMSE f_1	Bias±std. f_1	RMSE d_1	Bias±std. d_1
25dB	HTLS		0	0.00239	-0.00022 ± 0.00239	0.01489	-0.00214 ± 0.01477
	CA-HTLS		0	0.00228	-0.00014 ± 0.00228	0.01431	-0.00154 ± 0.01426
	MV-HTLS		0	0.00220	-0.00005 ± 0.00221	0.01360	0.00022 ± 0.01363
	STLS-HTLS	0	0	0.00208	-0.00013 ± 0.00208	0.01252	-0.00055 ± 0.01254
	HTLN	0	0	0.00208	-0.00013 ± 0.00208	0.01252	-0.00055 ± 0.01254
20dB	HTLS		11	0.00392	-0.00050 ± 0.00389	0.02714	0.00086 ± 0.02720
	CA-HTLS		9	0.00374	-0.00033 ± 0.00374	0.02877	-0.00053 ± 0.02884
	MV-HTLS		11	0.00352	-0.00028 ± 0.00352	0.02357	0.00390 ± 0.02330
	STLS-HTLS	0	3	0.00353	-0.00060 ± 0.00348	0.02420	0.00045 ± 0.02426
	HTLN	0	3	0.00355	-0.00056 ± 0.00352	0.02414	0.00046 ± 0.02419
15dB	HTLS		96	0.00506	-0.00027 ± 0.00508	0.05980	-0.00411 ± 0.05995
	CA-HTLS		62	0.00491	-0.00047 ± 0.00491	0.05517	-0.00239 ± 0.05532
	MV-HTLS		46	0.00464	-0.00033 ± 0.00464	0.04255	0.00469 ± 0.04243
	STLS-HTLS	12	71	0.00427	-0.00062 ± 0.00425	0.04157	0.00000 ± 0.04175
	HTLN	2	86	0.00453	-0.00043 ± 0.00453	0.04550	0.00185 ± 0.04567