CYCLOSTATIONARY SPECTRAL ANALYSIS OF SUBBAND ADAPTIVE FILTERS

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ABSTRACT

This paper presents cyclostationary spectral analysis of subband adaptive filters. First, the convergent point of the LMS type algorithm is determined. Next, using the spectral theory of cyclostationary processes, the cyclic spectral density matrix of the error signal is derived. Finally, its averaged variance is calculated for typical value of the delay in the desired signal and is compared with the simulation result.

1 INTRODUCTION

Recently, there have been many works concerning subband adaptive digital filters (ADF)[1, 2]. A block diagram for the typical 2-band case is shown in Figure 1. Advantages of the subband ADF over the conventional fullband ADF are as follows. Due to the decimation $\downarrow 2$ in the input signal part and the desired signal part the sampling rate is half of that of the fullband case and the filter lengths are also half with better convergence properties.

In [1] performance analysis of the subband ADF has been done by using the (noncausal) Wiener filter theory. Assuming that $H_0(z), H_1(z)$ are sufficiently close to ideal lowpass, highpass, respectively, it is shown in that the overall transfer function between the input and the output of the subband ADF in Figure 1 is a delayed version of that of $W(z)$ i.e., $W(z)z^{-m}$ where $W(z)$ is the transfer function of the fullband Wiener filter and m is the delay of the filter bank.

On the other hand, in [2] an alternative scheme in Figure 2 has been proposed. In [3] an analysis of the scheme in Figure 2 has been presented. Since the error signals are generated after interpolation $\uparrow 2$, the intermediate signals become cyclostationary with period 2. The spectral characteristics of the signals have been examined and for good lowpass and highpass filters the optimal filter $G_i(z)$ minimizing $E[e_i^2(n)] (i = 0, 1)$ is determined. The main result of [3] is that the overall transfer function of the system is approximately given by $W(z)$. Thus the scheme in Figure 2 does not have the delay in the scheme in Figure 1.

However, the analysis in [3] treats the ideal case where the analysis and the synthesis filters are close to ideal lowpass, highpass and the adaptive filters $G_i(z)$ are noncausal. In this paper we treat more realistic case where $G_i(z)$ are causal FIR with L taps and lowpass, highpass filters have overlapps.

First we determine the convergent point of the LMS type adaptive algorithm. Due to the second decimation in Figure 2 the error signals $e_i(n)$ become stationary again. So, using the statistics of jontly stationary $x(n)$ and $d(n)$, a system of $2L$ linear equations for $2L$ tap coefficients is first derivied. After solving this equation, we derive the variance expression of the overall error signal $e(n)$. However, this signal is no longer stationary, since the output $y(n)$ is cyclostationary with period 2. Using the method in [4, 5] we derive the cyclic spectral density matrix of $e(n)$ for the general M band case and its averaged variance. Some numerical results are presented to see the performance of the adaptive algorithms.

2 ANALYSIS OF THE SCHEME

First, we review some facts about stationary processes. Let $S_{xx}(z)$, $S_{dx}(z)$ be the spectral density of the zero mean stationary input signal $x(n)$, the cross spectral density between the zero mean stationary desired signal $d(n)$ and $x(n)$, respectively. That is,

$$S_{xx}(z) = \sum_{k=-\infty}^{\infty} R_{xx}(k)z^{-k},\ S_{dx}(z) = \sum_{k=-\infty}^{\infty} R_{dx}(k)z^{-k} \quad (1)$$

where $R_{xx}(k) = E[x(n+k)x(n)]$, $R_{dx}(k) = E[d(n+k)x(n)]$. These are evaluated on $z = e^{j\omega}$. It is well known that

$$S_{xx}(z) = \lim_{N\to\infty} \frac{1}{2N+1} E[|\sum_{n=-N}^{N} x(n)z^{-n}|^2] \quad (2)$$

$$S_{dx}(z) = \lim_{N\to\infty} \frac{1}{2N+1} E[(\sum_{n=-N}^{N} d(n)z^{-n}) \cdot (\sum_{n=-N}^{N} x(n)z^{-n})^*] \quad (3)$$

where $z = e^{j\omega}$ and $*$ denotes the complex conjugate. Also we note that for $\omega \neq \nu \pmod{2\pi}$

$$\lim_{N\to\infty} \frac{1}{2N+1} E[(\sum_{n=-N}^{N} z(n)e^{-j\omega n}) \cdot (\sum_{n=-N}^{N} x(n)e^{-j\nu n})^*] = 0 \quad (4)$$

where $z(n) = x(n)$ or $d(n)$.

Here we use the convention that the z-transform of a signal denoted by a small letter is expressed by the corresponding capital letter. As in [3], applying the z-transform relations to the signals in Figure 2, we formally have

$$Y(z) = X_0(z^2)G_0(z^2)F_0(z) + X_1(z^2)G_1(z^2)F_1(z) \quad (5)$$

$$E_i(z) = D_i(z) - \frac{1}{2}[H_i(z^{\frac{1}{2}})Y(z^{\frac{1}{2}}) + H_i(-z^{\frac{1}{2}})Y(-z^{\frac{1}{2}})] \quad (6)$$

with

$$D_i(z) = \frac{1}{2}[H_i(z^{\frac{1}{2}})D(z^{\frac{1}{2}}) + H_i(-z^{\frac{1}{2}})D(-z^{\frac{1}{2}})] \quad (i = 0, 1). \quad (7)$$

Substituting (5) into (6), we have

$$\begin{aligned} E_0(z) &= D_0(z) \\ &- \frac{1}{2}[H_0(z^{\frac{1}{2}})F_0(z^{\frac{1}{2}}) + H_0(-z^{\frac{1}{2}})F_0(-z^{\frac{1}{2}})]X_0(z)G_0(z) \\ &- \frac{1}{2}[H_0(z^{\frac{1}{2}})F_1(z^{\frac{1}{2}}) + H_0(-z^{\frac{1}{2}})F_1(-z^{\frac{1}{2}})]X_1(z)G_1(z) \quad (8) \end{aligned}$$

and a similar equation for $E_1(z)$. Assuming that $H_0(z), F_0(z)$ and $H_1(z), F_1(z)$ are sufficiently close to ideal lowpass and highpass, respectively, we approximate (8) as

$$E_i(z) \cong D_i(z) - M_i(z^{1/2})X_i(z)G_i(z) \quad (i = 0, 1) \quad (9)$$

with

$$M_i(z) = \frac{1}{2}[H_i(z)F_i(z) + H_i(-z)F_i(-z)] \quad (i = 0, 1). \quad (10)$$

The above z-transform relations are valid for finite energy signals. The stationary input signals $x_i(n)$ are not of finite energy. So we need to truncate the summations of z-transforms at $k = \pm N$ but by these truncations the relations such as in (5), (6) do not hold exactly due to the end effects at $k = \pm N$. However, these effects can be negligible for sufficiently large N. With this understanding we use the expression (9) to derive optimal subband filters $G_i(z)$ minimizing $E[e_i^2(n)]$.

These are given by

$$G_i(z) \cong \frac{S_{d_i x_i}(z)}{M_i(z^{\frac{1}{2}})S_{x_i x_i}(z)} \quad (i = 0, 1). \quad (11)$$

Even if the subband Wiener filter $S_{d_i x_i}(z)/S_{x_i x_i}(z)$ is close to be causal, since $M_i(z)$ is not minimum phase, $G_i(z)$ in (11) is noncausal. This point was not considered in [3]. Actually, it is seen for many pairs of $H_i(z), F_i(z)$ that $M_i(z)$ is close to a pure delay. Let us assume tentatively that

$$M_0(z) \cong z^{-m}. \quad (12)$$

To cancel $M_i(z^{1/2})$ in the denominator in (11), $d_i(n)$ is delayed by $m/2$ time units so that $d_i'(n) = d_i(n - m/2)$ and

$$S_{d_i' x_i}(z) = z^{-\frac{m}{2}} S_{d_i x_i}(z) \quad (13)$$

Hence, the corresponding optimal filter becomes the subband Wiener filter $S_{d_i x_i}(z)/S_{x_i x_i}(z)$. From Figure 2 $\downarrow 2$ operation precedes the signal $d_i(n)$, so that we need to delay the desired signal $d(n)$ by m time units. Thus the delay of the scheme in Figure 2 is m when we restrict that the filters $G_i(z)$ are causal. This is different from the result in [3] that there is no delay when the filters $G_i(z)$ are allowed to be noncausal.

Thus the LMS type algorithm for adapting the coefficient $g_i(k)$ of $G_i(z)$ $(i = 0, \ldots, L-1)$ is given by

$$g_i(k) \leftarrow g_i(k) + \mu x_i(n - k - \frac{m}{2})e_i'(n) \quad (0 < \mu \ll 1) \quad (14)$$

where $e_i'(n)$ is the error signal when the desired signal $d(n)$ is replaced by its delayed version $d(n-m)$. The convergent point of (14) is given by

$$E[x_i(n - k - \frac{m}{2})e_i'(n)] = 0 \quad (i = 0, 1;\ k = 0, \ldots, L-1) \quad (15)$$

Using the statistics of $d(n)$ and $x(n)$ with (8), we can write a system of $2L$ linear equations for $2L$ tap coefficients.

3 CYCLOSTATIONARY SPECTRAL ANALYSIS

Let us briefly review the facts about cyclostationary processes. A process $y(n)$ with zero mean is said to be cyclostationary with period M if

$$E[y(m)y(n)] := R(m, n) = R(m + M, n + M). \quad (16)$$

So for fixed u, $R(n+u, n)$ is a periodic sequence in n with period M. Thus we have the following discrete Fourier expansion:

$$R(n+u, n) = \sum_{k=0}^{M-1} c_k(u)W^{-kn} \quad (17)$$

where $W = e^{-j2\pi/M}$. Conversely, $c_k(u)$ is expressed as

$$c_k(u) = \frac{1}{M}\sum_{n=0}^{M-1} R(n+u, n)W^{kn}. \quad (18)$$

It is known that under certain conditions the sequence $c_k(u)$ has the following Fourier reprensentation:

$$c_k(u) = \frac{1}{2\pi}\int_0^{2\pi} F_k(\omega)e^{j\omega u}d\omega \quad (19)$$

$$F_k(\omega) = \sum_{u=-\infty}^{\infty} c_k(u)e^{-j\omega u} \quad (20)$$

These $F_k(\omega)$ are called the cyclic spectral density of a cyclostationary signal. It should be noted that if $y(n)$ is stationary, then $F_0(\omega)$ is the conventional spectral density and $F_k(\omega) = 0$ for $k = 1, \ldots, M-1$. Also we note that the averaged variance over one period

$$c_0(0) = \frac{1}{M}\sum_{n=0}^{M-1} R(n,n) \quad (21)$$

is given by

$$c_0(0) = \int_0^{2\pi} F_0(\omega)d\omega. \quad (22)$$

The process $\{y(n)\}$ itself has the representation

$$y(n) = \sum_{k=0}^{M-1} W^{-kn}z_k(n) \quad (23)$$

where $z_k(n)$ is the amplitude process of the output of the ideal bandpass filter whose center frequency and bandwidth are k/M and $1/M$, respectively. These $\{z_k(n)\}$ are jointly stationary with the cyclic spectral density matrix $\boldsymbol{F}(\omega) = \{F_{ik}(\omega)\}$. Moreover, for $|\omega| \le \omega_0/2 = \pi/M$

$$F_{ik}(\omega) = F_{i-k}(\omega + i\omega_0). \quad (24)$$

As in [4], let us define the following process

$$\boldsymbol{y}^T(n) = (y(n), W^n y(n), \ldots, W^{(M-1)n}y(n)). \quad (25)$$

Then we can show that

$$\boldsymbol{F}(\omega) = \lim_{N\to\infty}\frac{1}{2N+1}E[(\sum_{n=-N}^{N}\boldsymbol{y}(n)e^{-j\omega n}) \cdot(\sum_{n=-N}^{N}\boldsymbol{y}^T(n)e^{-j\omega n})^*]. \quad (26)$$

Now we derive the cyclic spectral density matrix $\boldsymbol{F}(\omega)$ of the error signal $e(n)$ in Figure 2. Here we treat the general M band case. The z-transform of $e(n)$ can be generalized as

$$E(z) = z^{-\Delta}D(z) - \sum_{l=0}^{M-1} A_l(z)X(W^l z) \quad (27)$$

where for generality, the delay in $d(n)$ is Δ and

$$A_l(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_k(W^l z)G_k(z^M)F_k(z). \quad (28)$$

From (26) we have

$$F_{ik}(\omega) = \lim_{N\to\infty}\frac{1}{2N+1}E[E(e^{j(\omega+i\omega_0)})E^*(e^{j(\omega+k\omega_0)})]. \quad (29)$$

Using (27) and noting (2), (3) and (4) we obtain

$$\begin{aligned} F_{ik}(\omega) = {} & \delta_{ik}e^{-j(i-k)\omega_0\Delta}S_{dd}(e^{j(\omega+i\omega_0)}) \\ & - e^{j(\omega+k\omega_0)\Delta}A_{i-k}(e^{j(\omega+i\omega_0)})S^*_{dx}(e^{j(\omega+k\omega_0)}) \\ & - e^{-j(\omega+i\omega_0)\Delta}A^*_{k-i}(e^{j(\omega+k\omega_0)})S_{dx}(e^{j(\omega+i\omega_0)}) \\ & + \sum_{l=0}^{M-1} A_l(e^{j(\omega+i\omega_0)})A^*_{l+k-i}(e^{j(\omega+k\omega_0)}) \\ & \cdot S_{xx}(e^{j(\omega+(i-l)\omega_0)}) \quad (i \ge k) \end{aligned} \quad (30)$$

where $F_{ki}(\omega) = F^*_{ik}(\omega)$ and $A_l(e^{j\omega}) = A_{l+M}(e^{j\omega})$ for $l < 0$. From this, we have

$$\begin{aligned} F_0(\omega) = {} & S_{dd}(e^{j\omega}) - e^{j\omega\Delta}A_0(e^{j\omega})S^*_{dx}(e^{j\omega}) \\ & - e^{-j\omega\Delta}A^*_0(e^{j\omega})S_{dx}(e^{j\omega}) \\ & + \sum_{l=0}^{M-1}|A_l(e^{j\omega})|^2 S_{xx}(e^{j(\omega-l\omega_0)}). \end{aligned} \quad (31)$$

In particular for $M = 2$, from (22) the averaged variance of $e(n)$ over one period is given by

$$\begin{aligned} \sigma^2 = {} & R_{dd}(0) - 2\sum_i a_i R_{dx}(i+\Delta) \\ & + \sum_i\sum_{i'} a_i a_{i'} R_{xx}(i-i') \\ & + \sum_i\sum_{i'} b_i b_{i'} R_{xx}(i-i') \end{aligned} \quad (32)$$

where $A_0(z) = \sum_i a_i z^{-i}$, $A_1(z) = \sum_i b_i z^{-i}$. The fourth term mainly represents the aliasing effect of the filter bank. Assuming that the unknown system is a 15th order FIR filter with SNR = 40dB, and using the CQF bank with 16 taps. Here the delay m in (12) is 15 but $m/2$ should be an integer, so in (14) we take $m = 16$. This may cause some performance loss. In Figure 3 for $\Delta = 15$ the empirical variance of the error signal is presented together with the theoretical value in (32) which is shown by a dotted line.

4 CONCLUSION

We have presented cyclostationary spectral analysis of subband adaptive filter. Through this study we feel that for the scheme in Figure 2 it is desirable to use a filter bank with odd taps.

References

[1] Y.G. Yang, N.I. Cho, and S.U. Lee, "On the performace analysis and applications of the subband adaptive digital filter," *Signal Processing*, vol. 41, pp. 295-307, 1995.

[2] M.R. Petraglia and S.K. Mitra, "Performance analysis of adaptive filter structure based on subband decomposition," *Proc. ISCAS*, pp. 60-63, 1993.

[3] H. Sakai and N. Hirayama, "Analysis of subband adaptive filters," *Proc. ICASSP*, 1996.

[4] H. Sakai, "On the spectral density matrix of a periodic ARMA process," *J. Time Ser. Anal.*, vol. 12, pp. 73-82, 1991.

[5] S. Ohno and H. Sakai, "Optimization of filter banks using cyclostationary spectral analysis," *Proc. ICASSP*, vol. 3, pp. 1292-1295, Detroit, 1995.

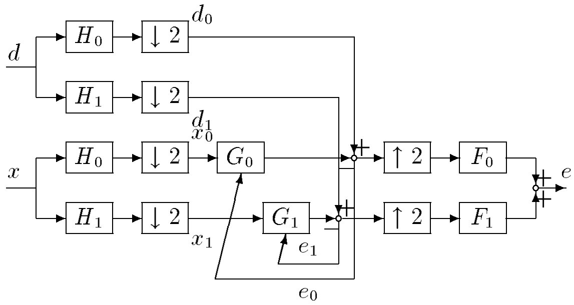

Figure 1: Conventional subband ADF

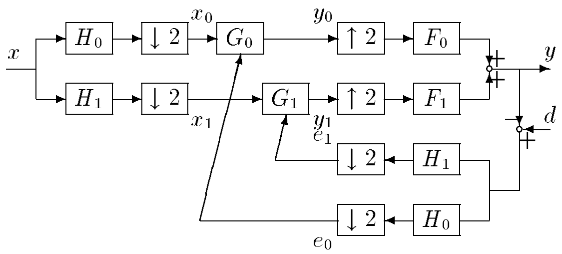

Figure 2: Alternative subband ADF

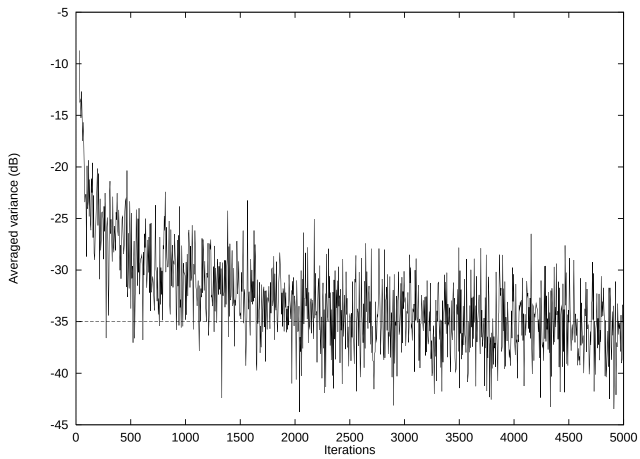

Figure 3: The empirical variance of the error signal