Extension of a Hyperstable Adaptive Line Enhancer for Tracking of Multiple Cisoids

Mukund PADMANABHAN and Petr TICHAVSKÝ†

IBM T.J. Watson Research Center, PO Box 218, Yorktown Heights, NY 10598, U.S.A., Phone: (914) 784 6985, E-Mail: mukund@watson.ibm.com

† *Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic Box 18, 182 08 Prague, Czech Republic, Fax:+42-2-6641 4702, E-Mail: tichavsk@utia.cas.cz*

Abstract. A hyperstable ALE for tracking complex cisoids is presented. The ALE incorporates an adaptive IIR filter, with the convergence of the filter being conditional on the overall system being 'passive'. The passivity of the system depends on the location of the input cisoid frequencies, and it is shown that for the case of upto two cisoids, the system is passive for all distinct frequencies. For the case of larger number of cisoids, the system is passive for certain ranges of the cisoid frequencies. Simulations are also given to back up the theoretical results.

I. INTRODUCTION

A hyperstable adaptive line enhancer for tracking real-valued noisy sinusoidal signals was introduced in [1, 2]. In this paper the algorithm is modified for processing of complex-valued signals and its convergence properties are discussed. The main advantage of the HALE algorithm compared to other algorithms multiple frequency tracker, [4], adaptive notch filter, [3, 4], is that the global convergence of the algorithm is *guaranteed* under a certain condition, which is always fulfilled for the one and two-cisoid case. Another advantage of HALE is its computational simplicity.

The signal under consideration has the form

$$y_n = \sum_{k=1}^{p} A_{kn} + v_n \qquad n = 0, 1, 2, \ldots \tag{1}$$

where A_{kn} represents the k-th cisoid at the time instant n, v_n is the noise and p is the (known) number of the cisoids. Assume that A_{kn} evolves according to

$$A_{kn} = e^{i\omega_{kn}} A_{k,n-1}(1 + h_{kn}) \qquad k = 1, \ldots, p. \tag{2}$$

In (2), h_{kn} is implicitly defined as the relative increment of the magnitude of the k-th cisoid and ω_{kn} is the k-th instantaneous angular frequency, given as the angle increment of A_{kn},

$$\omega_{kn} \triangleq \operatorname{Arg}\left[\frac{A_{kn}}{A_{k,n-1}}\right]. \tag{3}$$

The stationary case is given by (1) and (2) with constant $\{\omega_{kn}\}$ and $\{h_{kn} \equiv 0\}$.

The main problem to be solved is the recursive estimating (tracking) of the vector

$$\Omega_n = (\omega_{1n}, \ldots, \omega_{pn})^T. \tag{4}$$

In the algorithm this task is performed by tracking the vector parameter

$$\theta_n = (a_{1n}, \ldots, a_{pn})^T, \tag{5}$$

which is related to Ω_n by means of the polynomial

$$1 + D_n(z) \triangleq \prod_{k=1}^{p} (1 - e^{i\omega_{kn}} z^{-1}) = 1 + a_{1n} z^{-1} + \ldots + a_{pn} z^{-p}. \tag{6}$$

For a given θ_n, the frequencies ω_{kn} are given as angular positions of roots of the polynomial on the right-hand side of (6).

Like other frequency trackers, the HALE algorithm is represented by an update formula for θ_n using θ_{n-1} and the last available value of the signal, y_n.

The global convergence of the algorithm will be studied using a stationary and noiseless signal, i.e. ($v_n \equiv 0$) and $\Omega_n \equiv \bar{\Omega} = [\bar{\omega}_1, \ldots, \bar{\omega}_p] = \text{const.}$. All quantities related to the input signal will be denoted by bar, and if they are constant in time, the index n is deleted. In this notation the signal $\{\bar{y}_n\}$ obeys the recursion

$$\bar{y}_n = -\sum_{k=1}^{p} \bar{a}_k \bar{y}_{n-k}. \tag{7}$$

II. HALE ALGORITHM

The algorithm has two design variables: $g \in (0, 2/p)$ and $\mu \in (0, \infty)$. For the reasons explained later it is recommended to choose $g = 1/p$: with this choice, the tradeoff between the noise rejection and the tracking speed of the algorithm is controlled by the single parameter μ.

Besides the parameter θ_n, the algorithm uses an auxilinary complex-valued sequence $\{\varepsilon_n\}$. The algorithm is given by the recursions

$$\varepsilon_n = \frac{y_n - \theta_{n-1}^T X_n}{1 + \mu X_n^H X_n} \quad (8)$$

$$\theta_n = \theta_{n-1} + \mu X_n^* \varepsilon_n \quad (9)$$

where

$$X_n = \begin{bmatrix} -y_{n-1} + (1-g)\varepsilon_{n-1} \\ -y_{n-2} + (1-2g)\varepsilon_{n-2} \\ \vdots \\ -y_{n-p} + (1-pg)\varepsilon_{n-p} \end{bmatrix} \quad (p \times 1) \quad (10)$$

and the superscripts "T" and "H" denote the transpose and the conjugate transpose, respectively.

Nice properties of the above algorithm depend on the following function,

$$H(z) = \frac{1}{1 + g\bar{N}(z) + \bar{D}(z)} \quad (11)$$

where $\bar{D}(z)$ is defined as in (6), and

$$\bar{N}(z) = -(\bar{a}_1 z^{-1} + 2\bar{a}_2 z^{-2} \ldots + p\,\bar{a}_p z^{-p}). \quad (12)$$

Proposition 1 Assume that the function $H(z)$ in (11) is strictly positive real (SPR), (i.e. poles of $H(z)$ lie strictly in the unit circle and $\mathrm{Re}(H(z)) > 0$ for $|z| = 1$). Then, the HALE algorithm in (8)-(10) applied to the stationary noiseless signal $\{\bar{y}_n\}$ converges to the true parameters of the signal, i.e. $\theta_n \to \bar{\theta}$ and $\varepsilon_n \to 0$ as n goes to infinity for any $\mu \in (0, \infty)$ and for any initialization of the algorithm. In this case we say that the algorithm is hyperstable.

Proof of the proposition follows from the derivation of the algorithm in Appendix.

Note that the proof of convergence is given for the noiseless case (this is true for almost all hyperstable schemes). However, even in the noisy case, if the signal-to-noise ratio is sufficiently high, the algorithm stability in the noiseless case implies that the estimated parameter θ (or the frequencies) will converge to some neighborhood of the true parameter first, and then fluctuate around the true values.

III. CONDITION OF SPR OF $H(z)$

In this section we check the SPR property of $H(z)$.

It can be shown that the requirement that poles of $H(z)$ lie strictly inside the unit circle is assured by the condition $0 < g < 2/p$ [6, Section 2.2]. For $p = 1$ and $p = 2$, $H(z)$ becomes

$$H(z) = \frac{1}{1 - (1-g)\bar{z}_1 z^{-1}} \quad (13)$$

and

$$H(z) = \frac{1}{1 - (1-g)(\bar{z}_1 + \bar{z}_2)z^{-1} + (1-2g)\bar{z}_1\bar{z}_2 z^{-2}} \quad (14)$$

respectively, where $\bar{z}_k = e^{i\bar{\omega}_k}$, $k = 1, 2$.

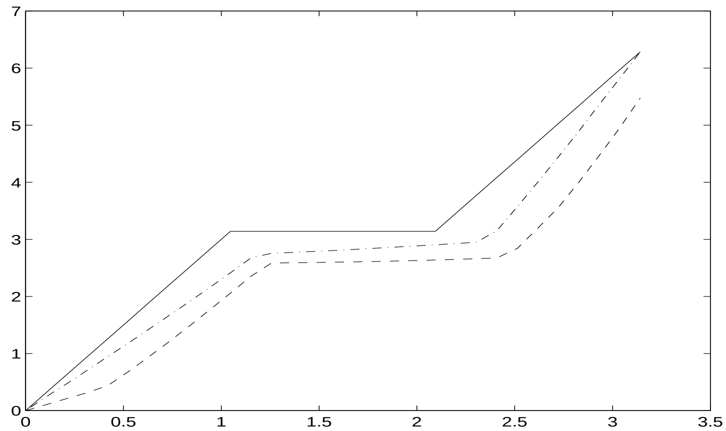

Fig. 1 Measure of the admissible set of ω_3 versus varying ω_2 for $\omega_1 = 0$. Full line for $g = 1/3$, dash-dotted line for $g = 0.5$ and dashed line for $g = 0.2$.

It can be easily seen that $H(z)$ is SPR for $p = 1$ and any $g \in (0, 2)$, and for $p = 2$, $g = 1/2$ and arbitrary distinct frequencies. In general, it appears that for $p > 2$, $H(z)$ may or may not be SPR, depending on g and on the frequencies. For the real-case [2], some conditions were given, that implied that $H(z)$ would be SPR if the input frequencies lay in a constrained space; so far, it has still not been possible to formulate similar conditions for the complex case.

In addition, our computer simulations like the one below show that for $g = 1/p$ the function $H(z)$ is SPR on the largest range of the frequencies. Usually, $H(z)$ is SPR if the frequencies are well separated.

Example 1 Let $p = 3$, $\bar{\omega}_1 = 0$, $\bar{\omega}_2$ is varying in the interval $(0, \pi)$ and consider measure of the set of $\bar{\omega}_3$ such that the corresponding function $H(z)$ is SPR. For brevity, we call the set of such $\bar{\omega}_3$ the admissible set. The simulation shows that the set is composed of 1-3 intervals.) The results for $g = 0.2$, $g = 1/3$ and $g = 0.5$ are shown in Figure 1.

We observe that if $\bar{\omega}_2$ is close to $\bar{\omega}_1$, the interval of admissible $\bar{\omega}_3$ is narrow, its measure is close to 0. If $\bar{\omega}_2$ is close to $\bar{\omega}_1 + \pi$, almost the whole interval $(-\pi, \pi)$ is admissible, perhaps except for narrow intervals around $\bar{\omega}_1$ and $\bar{\omega}_2$.

It has been found empirically that the SPR condition for $p = 3$ and $g = 1/3$ is equivalent to

$$\min_{j,k} |\bar{\omega}_j - \bar{\omega}_k|_o + \max_{j,k} |\bar{\omega}_j - \bar{\omega}_k|_o > \pi. \quad (15)$$

Here, $|\bar{\omega}_j - \bar{\omega}_k|_o$ means the frequency difference modulo 2π, i.e. a number between 0 and π.

III. SIMULATIONS

Example 2 Tracking time for various μ. Consider a model with two stationary noiseless cisoids with constant unit magnitudes and well separated frequencies,

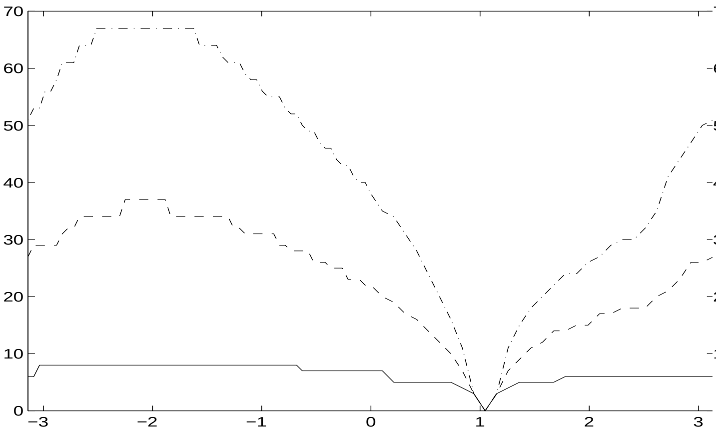

Fig. 2 Tracking time of two stationary frequencies $\omega_{1,2} = \pm\pi/3$ versus initial estimate of ω_2 for $\mu = 0.05$ (dashed-dotted line), 0.1 (dashed line) and 1 (solid line).

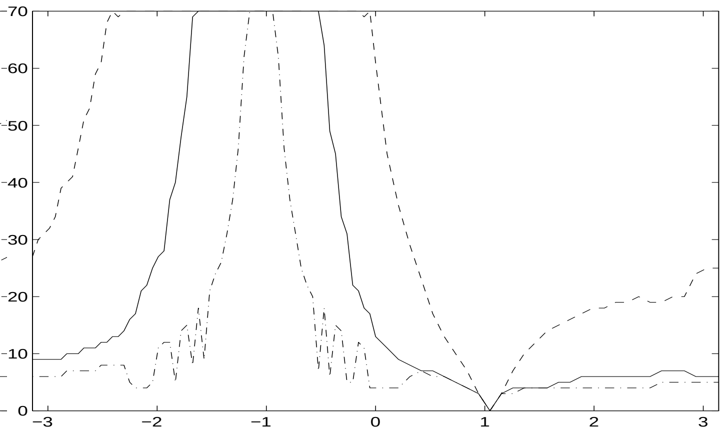

Fig. 3 Tracking time of two frequencies with $\bar{\omega}_1 = -\pi/3$, $\mu = 0.1$ (dashed line), 1 (solid line), and 100 (dashed-dotted line), $g = 1/2$, versus $\bar{\omega}_2$ ($\bar{\omega}_2 \in (-\pi, \pi)$).

$\bar{\omega}_1 = -\pi/3$, $\bar{\omega}_2 = \pi/3$. The algorithm is initialized with these frequencies: the first one is correct, equal to $\bar{\omega}_1$, and the other one is wrong and varying in the range $(-\pi, \pi)$. Starting from these frequencies, the time in which both the estimated frequencies fall into the ± 0.1 tolerance interval around the true frequencies is measured. The result for various parameters μ is shown in Figure 2.

Example 3 Tracking time versus varying frequency. In this experiment, the input has two noiseless cisoids; the first one at frequency $-\pi/3$, and the frequency of the second cisoid is varied over the interval $(-\pi, \pi)$. The algorithm is initialized at the frequencies $\pm\pi/3$, and allowed to track the input. The time in which both estimated frequencies fall within a ± 0.1 tolerance interval around the true frequency was measured, and the result is shown in Figure 3, for μ values of 0.1, 1 and 100. The tracking time can be seen to degrade as the spacing between the two cisoid frequencies reduces; however, the system is still guaranteed to converge to the correct values if the two input frequencies are distinct.

IV. CONCLUSIONS

The hyperstable ALE formulated in [2] was extended to deal with complex cisoids. The convergence of the filter in [2] is conditional on the system being hyperstable, which in turn implies that the underlying system is passive, and excited with finite energy inputs. Though these results applied for the case of real-valued signals, the underlying physical implication of passivity and finite energy can be extended to the case of complex signals also – leading to the complex HALE formulation of this paper.

The passivity condition is dependent on the location of the input cisoid frequencies; for the case of upto two cisoids, the passivity condition is satisfied for all distinct frequencies, for the case of larger number of cisoids, the passivity condition is satisfied only for certain ranges of the cisoid frequencies. Examples are given to illustrate this. Finally, simulations are given to illustrate the convergence of the algorithm. These simulations indicate that the performance of the algorithm is robust in the presence of noise also.

V. APPENDIX: Algorithm Derivation

The derivation of complex version of the HALE algorithm follows that in [2]. The basic idea is to built a system of local resonators whose frequencies are equal to the estimated frequencies, and that generate a signal $\hat{y}_n$, which is an estimate of the input signal. The error between this local estimate and the actual input, $\{e_n\}$, $e_n = y_n - \hat{y}_n$, is used to adapt the frequencies, and states, of the resonators, to minimize the prediction error. The transfer function of the filtering function that represents the sum of the resonators is

$$H_{\text{lg}}^{(n)}(z) \triangleq g \sum_{k=1}^{p} \frac{e^{i\omega_{kn}} z^{-1}}{1 - e^{i\omega_{kn}} z^{-1}} = g \, \frac{N_n(z)}{1 + D_n(z)} \quad (16)$$

where $D_n(z)$ is defined in (6) and

$$N_n(z) = n_{1n} z^{-1} + \ldots, + n_{pn} z^{-p} \ . \quad (17)$$

Further, the coefficients of $N_n(z)$ and $D_n(z)$ are related as

$$n_{kn} = -k a_{kn} \ , \qquad k = 1, \ldots, p \quad (18)$$

(this can be easily proven by induction). The difference equations corresponding to (16) can be written as

$$\hat{y}_n \ = -\sum_{k=1}^{p} a_{kn} \hat{y}_{n-k} + g \sum_{k=1}^{p} n_{kn} e_{n-k} \quad (19)$$

$$e_n \ = y_n + \sum_{k=1}^{p} a_{kn} \hat{y}_{n-k} - g \sum_{k=1}^{p} n_{kn} e_{n-k}. \quad (20)$$

Now, assume that the input y_n is the stationary and noiseless signal $y_n = \bar{y}_n$. Using (7) and (20) we obtain

$$e_n + \sum_{k=1}^{p} (\bar{a}_k + g\bar{n}_k) e_{n-k} = -w_n \triangleq$$

$$\sum_{k=1}^{p}(a_{kn}-\bar{a}_k)\hat{y}_{n-k} - g\sum_{k=1}^{p}(n_{kn}-\bar{n}_k)e_{n-k}, \quad (21)$$

or, equivalently,

$$e_n = H(z)[-w_n] \quad (22)$$

where

$$w_n = (\theta_n - \bar{\theta})^T X_n, \quad (23)$$

and $H(z)$ and X_n were defined in (11) and (10), respectively. Hence, the input to $H(z)$ is the signal $-w_n$, and the output of the filter is e_n, and w_n is related to e_n through some non-linearity, and both w_n and e_n are complex.

In [5, 2], the theory of hyperstability was used to obtain conditions under which the error e_n would go to 0. This was essentially based on the following argument: if $H(z)$ is an SPR transfer function, it can be thought of (after applying the inverse bilinear transform, which preserves the SPR property in the analog domain also) as the impedance of a one-port network made up of lossy (resistances) and lossless (inductances, capacitances, transformers, gyrators) components, with the input to the transfer function being the current, and the output being the voltage. If the energy applied to such a circuit is finite, then the states of the circuit will die away to zero in time because the circuit comprises of only lossy and lossless elements. The energy going into the circuit is of course given by the integral of the product of the current and voltage, i.e. the integral of the product of the input and output signals of the transfer function. When translated into the digital domain, this is equivalent to saying that the sum over time, of the product of the input and output of $H(z)$, is upper bounded by some finite quantity.

These arguments can be carried over to the case of complex-valued signals also. Hence, if $H(z)$ is SPR, and if the 'real' energy going into the circuit is finite, then the states of the circuit will die to zero in time, which corresponds to the desired condition of convergence. As the input and output of $H(z)$ are complex-valued, the equivalent of the 'real' energy in the digital domain is $Re\{w_n e_n^*\}$, hence the condition of finite energy can be expressed as

$$\text{Re}\left\{\sum_{k=1}^{k_1} w_k e_k^*\right\} \geq -\gamma^2 \quad \forall k_1, \quad \gamma > 0. \quad (24)$$

As in [2] this condition can be satisfied simply by defining the update for θ_n as in (9).

It remains to rewrite the recursion (20) for e_n in a causally implementable form. Using the notation (10) we get

$$e_n = y_n - \theta_n^T X_n\,. \quad (25)$$

Substituting (9) into (25) and rearranging terms we obtain (8), as desired.

References

[1] M. Padmanabhan and K. Martin, "A second-order hyperstable adaptive filter for frequency estimation", *IEEE Trans. on Circuits and Systems–II: Analog and Digital Signal Processing*, Vol. 40, No. 6, June 1993, pp.398–403.

[2] M. Padmanabhan "A hyperstable adaptive line enhancer for fast tracking of sinusoidal inputs", *IEEE Trans. on Circuits and Systems–II: Analog and Digital Signal Processing*, Apr 1996.

[3] A. Nehorai, "A minimal parameter adaptive notch filter with constrained poles and zeros", *IEEE Trans. on Acoustics, Speech and Signal Processing*, Vol. ASSP-33, pp. 983–996, August 1985.

[4] P.Tichavský and P. Händel, "Two algorithms for adaptive retrieval of slowly time-varying multiple cisoids in noise", *IEEE Trans. on Signal Processing*, vol. 43, no.5, pp. 1116–1127, May 1995.

[5] C. Richard Johnson Jr., "A convergence proof for a hyperstable adaptive recursive filter", *IEEE Tr. Info. Theory*, pp. 745-749, vol. IT-25, Nov. 1979.

[6] M. Padmanabhan and K. Martin, "Resonator-Based Filter-Banks for frequency-domain Applications", *IEEE Trans. on Circuits and Systems*, vol. 38, no. 10, pp 1145–1159, Oct 1991.