NEW GEOMETRICAL RESULTS ABOUT 4-TH ORDER DIRECTION FINDING METHODS PERFORMANCE

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ABSTRACT

Since a decade, higher order direction finding (DF) methods have been developed for non gaussian signals. However, relatively few papers have been devoted to the performance analysis of these methods. The purpose of this paper is to present a geometrical analysis of the potential performance of these methods trough the new concept of "equivalent array", which makes possible the prediction of some of their performance.

1. INTRODUCTION

Up to the middle of the eighty, the DF methods exploit only the information contained in the second order statistics of the observations. However, since a decade, DF methods exploiting the information contained in the fourth order statistics of the data have been developed for non gaussian signals. Most of these techniques are fourth order extensions of second order techniques such as the 4-th MUSIC [1-2] or the 4-th ESPRIT methods [3], although a new concept of higher order DF techniques has been developed recently [4]. However, although promising for some applications, relatively few papers have been devoted to the performance analysis of these 4-th order DF methods. Among these few papers, we find in particular reference [1-2] and [5] which present either analytic or simulations results about the performance of the 4-th MUSIC method.

The purpose of this paper is to present a geometrical analysis of the potential performance of 4-th order DF methods trough the concept of "equivalent array". This new concept introduces physical interpretations of rather abstracted higher order algebraic results and makes possible the prediction of the potential resolution of 4-th order DF methods or the number of independent sources that can be processed by such techniques.

2. HYPOTHESES

We consider, in this paper, an array of N narrow-band (NB) omnidirectional sensors and we call $\boldsymbol{x}(t)$ the vector of the complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of P stationary sources corrupted by a noise. Under these assumptions, the observation vector can be written as follows

$$\boldsymbol{x}(t) = \sum_{i=1}^{P} m_i(t)\, \boldsymbol{a}(\theta_i) + \boldsymbol{b}(t) \overset{\Delta}{=} A\, \boldsymbol{m}(t) + \boldsymbol{b}(t) \qquad (2.1)$$

where $\boldsymbol{b}(t)$ is the noise vector, $\boldsymbol{m}(t)$ is the vector which components $m_i(t)$ are the complex amplitudes of the sources, A is the $(N \xi P)$ matrix of the sources steering vectors $\boldsymbol{a}(\theta_i)$, which contains the information about the direction of arrival θ_i of the sources. In particular, the component n of the vector $\boldsymbol{a}(\theta_i)$, noted $a_n(\theta_i)$, can be written, for a plane array, as

$$a_n(\theta_i) = \exp\{j2\pi[x_n \cos(\theta_i) + y_n \sin(\theta_i)] / \lambda\} \qquad (2.2)$$

where λ is the wavelength and where x_n and y_n are the coordinate of the sensor n of the array.

The second and fourth order DF methods exploit the information contained in the correlation matrix $R_x \overset{\Delta}{=} \mathrm{E}[\boldsymbol{x}(t)\boldsymbol{x}(t)^\dagger]$ and the quadricovariance Q_x (matrix of the 4-th order cumulants $\mathrm{Cum}[x_i(t), x_j(t)^*, x_k(t)^*, x_l(t)]$, where * means conjugation) of the data respectively. In the presence of a gaussian spatially white noise, the matrix R_x and Q_x are defined respectively by

$$R_x = \sum_{i,j=1}^{P} R_m(i, j)\, \boldsymbol{a}(\theta_i)\, \boldsymbol{a}(\theta_j)^\dagger + \eta_2 \mathrm{I} \qquad (2.3)$$

$$Q_x = \sum_{i,j,k,l=1}^{P} Q_m(i,j,k,l)[\boldsymbol{a}(\theta_i) \otimes \boldsymbol{a}(\theta_j)^*][\boldsymbol{a}(\theta_k) \otimes \boldsymbol{a}(\theta_l)^*]^\dagger \qquad (2.4)$$

where η_2 is the mean power of the noise per sensor, R_m and Q_m are the correlation matrix and the quadricovariance of the vector $\boldsymbol{m}(t)$ respectively and where $\otimes$ is the Kronecker product.

3. EQUIVALENT ARRAY CONCEPT FOR INDEPENDENT SOURCES

3.1 Presentation

In the presence of P independent sources, the expressions (2.3) and (2.4) become respectively

$$R_x = \sum_{i=1}^{P} R_m(i, i)\, \boldsymbol{a}(\theta_i)\, \boldsymbol{a}(\theta_i)^{\dagger} + \eta_2 \, \mathrm{I} \qquad (3.1)$$

$$Q_x = \sum_{i=1}^{P} Q_m(i, i, i, i)\, [\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]\, [\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]^{\dagger} \qquad (3.2)$$

In these conditions, the matrices Q_x and R_x have the same algebraic structure, where the auto-cumulant $Q_m(i, i, i, i)$ and the vector $[\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]$ play, for Q_x, the rule playing for R_x by the power $R_m(i, i)$ and the steering vector $\boldsymbol{a}(\theta_i)$ respectively. Thus, for the 4-th order DF methods, the $(N^2 \xi 1)$ vector $[\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]$ can be viewed as the *equivalent steering vector* of the source i for the *true array* of N sensors with coordinates (x_n, y_n), $1 \leq n \leq N$. Noting $m = r + N(q-1)$, where $1 \leq r, q \leq N$, the component m of the vector $[\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]$, noted $[\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]_m$ or $[\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]_{r,q}$, can be written as

$$[\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]_{r,q} = \qquad (3.3)$$

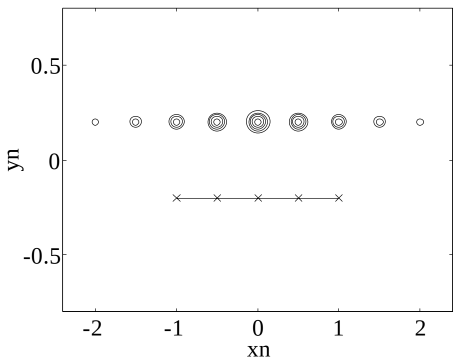
$$\exp\{j2\pi[(x_r - x_q)\cos(\theta_i) + (y_r - y_q)\sin(\theta_i)] / \lambda\}$$

Comparing (3.3) to (2.2), we deduce that the vector $[\boldsymbol{a}(\theta_i)\otimes\boldsymbol{a}(\theta_i)^*]$ can also be viewed as the *true steering vector* of the source i for the *equivalent array* (EA) of N^2 *equivalent sensors* (ES) with coordinates given by $(x_r - x_q, y_r - y_q)$, $1 \leq r, q \leq N$. Note that the N couples (r, q) such that $r = q$, $1 \leq r \leq N$, give the same ES of coordinates (0, 0). We will say that this ES is of multiplicity N and we deduce, from the previous results, that an array of N omnidirectional NB sensors admits, for the 4-th order DF problem, an EA of at most $N^2 - N + 1$ different ES.

The concept of EA is very useful to predict some of the 4-th order DF methods performance. In particular, if we note Ne the number of different sensors of the EA, the maximum number of independent sources that can be processed by a 4-th order DF method is, of course, $Ne - 1$, which is at most equal to $N^2 - N$. On the other hand, the aperture of the EA jointly with the multiplicity order of each sensor give information about the potential resolution of 4-th order DF methods through the beamwidth of the EA pattern. Some examples are presented in the following sections.

3.2 The linear array example

For a linear array, it is always possible to choose a coordinate system in which the sensor n has the coordinates $(x_n, 0)$, $1 \leq n \leq N$. As a consequence, the ES of the EA are at coordinates $(x_r - x_q, 0)$, $1 \leq r, q \leq N$, which also corresponds to a linear array.



o : Equivalent Array x : Real Array

Fig.1 - *EA of a ULA of 5 sensors, d = λ/2*

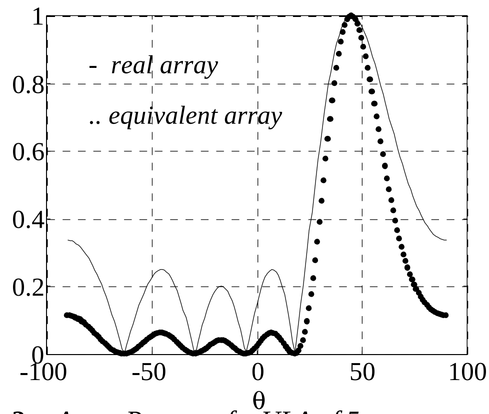


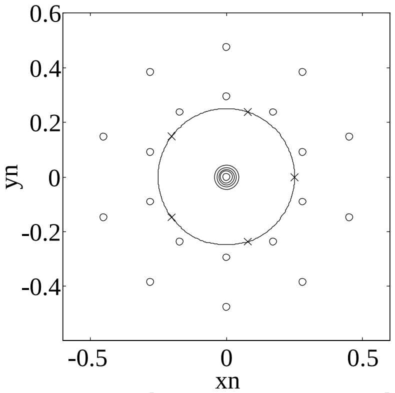
Fig.2 - *Array Pattern of a ULA of 5 sensors and its EA of 9 sensors, d = λ/2*

For a Uniformly spaced Linear Array (ULA), $x_n = (n-1)\, d$, where d is the interelement spacing, and the EA is composed of the sensors which coordinates are given by $x_{r,q} \overset{\Delta}{=} x_r - x_q = (r-q)d$, $1 \leq r, q \leq N$, which also corresponds to a ULA. More precisely, the EA has $Ne = 2N - 1$ different sensors, which means in particular that with a ULA of N sensors, the 4-th order DF methods are able to process $Ne - 1 = 2(N - 1)$ independent sources, result already found in [1] by algebraic considerations and verified in section 5. On the other hand, the sensor of the EA at the coordinate $x_{r,q}$ has a multiplicity of order $N - |r - q|$, which is in fact equivalent to a sensor which is weighted in amplitude by the factor $N - |r - q|$.

Thus the EA of a ULA of N sensors is an amplitude tappered ULA of $2N - 1$ sensors. This amplitude tappering explains in particular the reason why the beamwidth of the EA is not twice as narrow as that of the initial array despite of the fact that the physical size of the EA is twice greater as that of the initial array. All these results are illustrated by figures 1 and 2 which show the EA of a ULA of 5 sensors and the associated patterns respectively.

3.3 The circular array example

For a Uniformly spaced Circular Array (UCA) of N sensors, it is always possible to choose a coordinate system in which the sensor n has the coordinates $(R\cos\phi_n, R\sin\phi_n)$, $1 \le n \le N$, where R is the radius of the array and where $\phi_n = (n - 1)\, 2\pi / N$. The ES of the EA are at coordinates $(R_{rq}\cos\phi_{rq}, R_{rq}\sin\phi_{rq})$, $1 \le r, q \le N$, where $R_{rq} = 2R\sin[(r - q)\pi / N]$ and $\phi_{rq} = (r + q - 2 + N/2)\pi / N$.



o : Equivalent Array x : Real Array

Fig.3 - *EA of a UCA of 5 sensors, R = λ/2*

It is then easy to show that the ES which are not at coordinates (0, 0) lies on $(N - 1)/2$ or $N/2$ different circles if N is odd or even respectively. Moreover, for odd values of N, $2N$ ES lie on each circle of the EA, uniformly spaced and with an order of multiplicity equal to 1. We deduce from this result that the EA of a UCA of N odd sensors has $Ne = N^2 - N + 1$ different sensors, which means in particular that with a UCA of N odd sensors, the 4-th order DF methods are able to process $Ne - 1 = N^2 - N$ independent sources, as it is verified in section 5. The previous results are illustrated by figures 3 and 4 which show the EA of a UCA of 5 sensors and the associated patterns respectively.

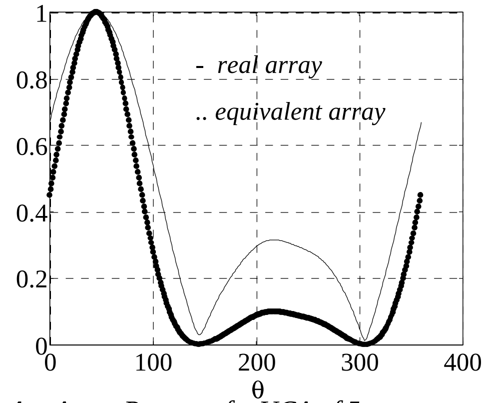


Fig.4 - *Array Pattern of a UCA of 5 sensors and its EA of 21 sensors, R = λ/2*

4. THE 4-TH ORDER CORRELATED SOURCES CASE

In the presence of 4-th order correlated sources, the quantity $Q_m(i,j,k,l)$ may not be equal to 0 when the indices i, j, k and l are not equal and it can be seen from (2.4) that, in this case, the rank of the quadricovariance matrix Q_x becomes generally greater than the number of sources P and is equal to P^2 when all the sources are 4-th order correlated to each other. This result is different from that obtained with second order statistics for which the rank of the correlation matrix (2.3) in the absence of noise remains equal to P as long as the sources are not perfectly coherent. For this reason, in the presence of 4-th order correlated sources, the matrix Q_x and R_x have not the same algebraic structure, and the concept of EA can no longer be applied. In these cases, the number of sources that can be processed by 4-th order methods depends on the 4-th order correlation between sources. In the worst case, all the sources are 4-th order correlated and the condition $P^2 < N^2$ must be verified, i.e $P \le N - 1$, which means that only $N - 1$ sources are able to be processed. In the best case, all the sources are statistically independent, and $Ne - 1$ sources can be processed.

Consequently, in the general case where only some of the sources are 4-th order correlated, the number of sources P that are able to be processed, for an array of N sensors, by a 4-th order DF method is such that

$$N - 1 \leq P \leq Ne - 1 \quad (4.1)$$

which is always greater than or equal to the number of sources that can be processed by second order methods and which is upper-bounded by the number of sensors of the EA, defined for independent sources, minus one.

5. PERFORMANCE ILLUSTRATIONS OF 4-TH MUSIC METHOD

To verify the results presented in the previous sections, we analyse, for different scenari of sources, the performance of the 4-th order MUSIC method, described in [1-2], for a ULA and a UCA of 3 sensors respectively, assuming that the statistics of the datas are perfectly known. Note that in this case, $Ne = 5$ for the ULA whereas $Ne = 7$ for the UCA.

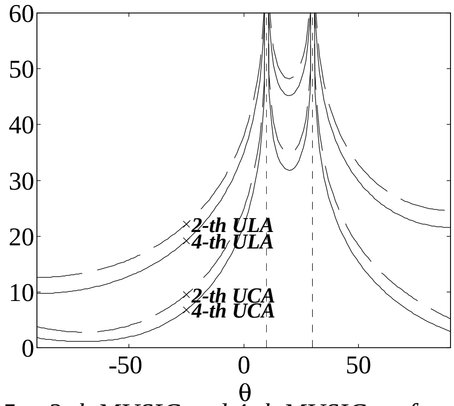


Fig.5 - *2-th MUSIC and 4-th MUSIC performance ULA and UCA, N = 3, P = 2, correlated sources*

The figure 5 shows the performance of 2-th and 4-th MUSIC in the presence of two 2-th and 4-th order correlated sources, both for the ULA and the UCA. Note the success of both techniques in finding the direction of the sources and the better resolution of the 4-th order method. The figures 6 shows the behaviour of 4-th order MUSIC, both for the ULA and the UCA array, in the presence 6 statistically independent sources respectively. Note the good performance of the method while $P \leq Ne - 1$ and the poor performance as soon as $P > Ne - 1$.

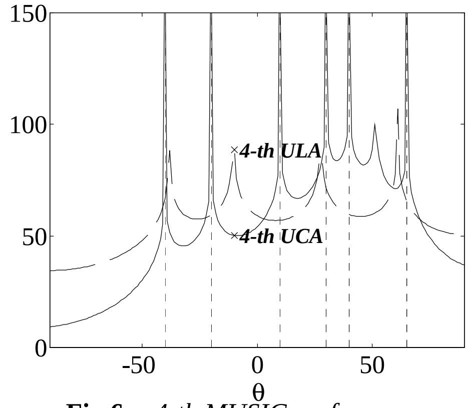


Fig.6 - *4-th MUSIC performance ULA and UCA, N = 3, P = 6 independent sources*

6.CONCLUSION

In this paper, the concept of "Equivalent Array" has been introduced to analyse the potential performance of all the 4-th order DF methods. This new concept is very useful to predict the potential resolution of all the 4-th DF methods and the maximal number of sources that can be processed by such techniques for a given array of sensors. For a given number of sensors, the array geometry has been shown to be a crucial parameter in the processing capacity of 4-th order techniques and the concept of Equivalent Array might be a useful tool for the optimization of this parameter.

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