ON BLIND IDENTIFICATION OF QUADRATIC SYSTEMS

Panos Koukoulas Nicholas Kalouptsidis

University of Athens
Department of Informatics
Division of Communications and Signal Processing
T.Y.P.A. Buildings, 157 71 Athens, Greece
koukoula@di.uoa.gr, kalou@di.uoa.gr

ABSTRACT

In this paper the blind identification problem of a finite extent quadratic system driven by a sequence of independent and identically distributed random variables is considered. Output cumulants up to the fourth-order are used and solutions are obtained for special cases of quadratic systems.

1 INTRODUCTION

We will be concerned with quadratic Volterra systems of the form

$$y(n) = \sum_{k_1=0}^{p} \sum_{k_2=0}^{p} h(k_1,k_2)u(n-k_1)u(n-k_2) \qquad (1)$$

The Volterra kernel $h(k_1,k_2)$ is a causal, symmetric sequence. The input signal $u(n)$ is a sequence of independent identically distributed random variables. Blind identification is concerned with the estimation of $h(k_1,k_2)$ on the basis of output statistics only.

Previous attempts concerned with quadratic blind identification are described in [1]- [2]. The input in [1] is assumed gaussian. This leads to considerable simplifications because the input cumulants greater than two are zero. The parameter set was chosen to be the one that minimizes the sum of the squared differences between the observed second and third order cumulants and the cumulants of the proposed model. The solution in general is not unique. The work of [2] addresses the problem of order determination and parameter estimation using general i.i.d. input and third order output cumulants. Under some conditions all the nondiagonal coefficients can be expressed in terms of the diagonal parameters and some third-order output cumulants. The diagonal parameters are solutions of nonlinear equations that can be found if a second output of the system resulting from a white gaussian sequence is available.

In this paper blind identification of (1) is addressed using cumulants of order 4 rather than 3. However the difficulties arising with the diagonal entries remain. To cope with this problem additional assumptions are imposed. The same results cannot be derived with information up to third order cumulants as in [2]. The main obstacle is the determination of the input cumulants.

Recall that if $x(k)$ is a stationary discrete time random process then the $p-th$ order cumulant of $x(k)$, denoted $c_{p,x}(k_1,k_2,\cdots,k_{p-1})$, is defined as the joint $p-th$ order cumulant of the random variables $x(k)$, $x(k+k_1),\cdots,x(k+k_{p-1})$, i.e.,

$$c_{p,x}(k_1,k_2,\cdots,k_{p-1}) = \mathrm{cum}(x(k),x(k+k_1),\cdots,x(k+k_{p-1}))$$

Because of stationarity, the $p-th$ order cumulant is a function of the $(p-1)$ lags $k_1,k_2,\cdots,k_{p-1}$. The function $c_{p,x}(.)$ is symmetric and is completely determined in the space Z^{p-1} by its values in the domain Δ defined by

$$\Delta = \{k_1,\cdots,k_{p-1}) \in Z^{p-1} : 0 \le k_{p-1} \le \cdots \le k_1\}$$

For every $l \in N$, define

$$\Delta_l = \{k_1,\cdots,k_{p-1}) \in \Delta : k_1 > l\}$$

The $p-th$ order polyspectrum is defined as the $(p-1)$-dimensional Fourier transform of the $p-th$ order cumulant $c_{p,x}(k_1,k_2,\cdots,k_{p-1})$. If $x(k)$ is an i.i.d sequence then $C_{p,x}(.) = \gamma_{p,x}$, where $\gamma_{p,x}$ is the p-th order cumulant of $x(k)$. The p-th order cumulant of the output of a second order system depends on all input cumulants of order less than $2p$.

Let us now introduce the LTI filters with impulse responses

$$h_k(l) = h(k+l,l) \quad , \qquad k,l \in Z$$

Then eq. (1) is written

$$y(n) = y_0(n) + 2\sum_{k>0} y_k(n)$$

where

$$y_k(n) = \sum h_k(l)x_k(n-l)$$

and

$$x_k(n) = u(n-k)u(n)$$

Certain output cumulants up to the $4-th$ order are next derived. The derivation relies on Leonov-Shiryaev formula [3] and is omitted due to space limitations.

$$c_{1,y} = \gamma_{2,u} \sum_n h_0(n) \qquad (2)$$

$$c_{2,y}(i) = \gamma_{2,u^2} \sum_n h_0(n)h_0(i+n) + 4\gamma_{2,u}^2 \sum_{k>0,n} h_k(n)h_k(i+n) \qquad (3)$$

where

$$\gamma_{2,u^2} = \gamma_{4,u} + 2\gamma_{2,u}^2$$

$$c_{3,y}(i_1,i_2) = \gamma_{6,u}\phi_0(i_1,i_2) + \gamma_{4,u}\gamma_{2,u}\phi_1(i_1,i_2) + \gamma_{3,u}^2\phi_2(i_1,i_2) + \gamma_{2,u}^3\phi_3(i_1,i_2) \qquad (4)$$

where the functions $\phi_i(.)$ are defined by

$$\phi_0(i_1,i_2) = \sum_n h_0(i_1+n)h_0(i_2+n)h_0(n) \qquad (5)$$

$$\phi_1(i_1,i_2) = \omega_1(i_1,i_2) + \omega_1(i_1-i_2,-i_2) + \omega_1(i_2-i_1,-i_1) \qquad (6)$$

with

$$\omega_1(i_1,i_2) = 4\sum_{k,n} h_k(i_1+n)h_k(i_2+n)h_0(n)$$

$$\phi_2(i_1,i_2) = \psi_2(i_1,i_2) + \omega_2(i_1,i_2) + \omega_2(i_1-i_2,-i_2) + \omega_2(i_2-i_1,-i_1) \qquad (7)$$

with

$$\psi_2(i_1,i_2) = 4\sum_{k,n} h_k(i_1+n)h_k(i_2+n)h_k(n)$$

$$\omega_2(i_1,i_2) = 2\sum_{k,n} h_0(i_1+k)h_0(i_2+n)h(n,k)$$

$$\phi_3(i_1,i_2) = 8\sum_{k,n,l} h(n,k)h(i_1+k,i_1+l) h(i_2+l,i_2+n) \qquad (8)$$

Since the Volterra kernel has finite extent of order p, the third order cumulant has only one nonvanishing term in Δ_p, [2]. This is given by

$$c_{3,y}(i_1,i_2) = \gamma_{3,u}^2\omega_2(i_1-i_2,-i_2) = 2\gamma_{3,u}^2 \sum_{k,n} h_0(i_1-i_2+k)h_0(n-i_2)h(n,k)$$

For the rest of the paper we assume that the order of the diagonal filter is p. Hence $h_0(0)h_0(p) \neq 0$. The fourth order cumulant has too many terms. Due to space limitation only the nonvanishing term in the domain Δ_{2p} is presented. The following formula will be instrumental for our purposes.

$$c_{4,y}(i_1,i_2,i_3) = 4\gamma_{2,u}\gamma_{3,u}^2 \sum_{k,n,l} h_0(i_1-i_2+k) h(k,i_2-i_3+l)h(l,n)h_0(l-i_3) \qquad (9)$$

Using third order cumulants information in the domain Δ_p, Bondon and Krob have shown that all the non diagonals terms can be expressed in terms of the diagonal terms and third order cumulants by the relation

$$C = 2\gamma_{3,u}^2 Hh \qquad (10)$$

where H is a lower triangular matrix whose diagonal elements are equal to $h_0(0)h_0(p)$ and the remaining entries are products of diagonal terms. h is a vector containing the nondiagonal terms. The vector C contains third order cumulants information.

Next with the information of first, second, third and fourth order cumulants of the output and the assumption that $\gamma_{3,u} \neq 0$ and $h_0(0)h_0(p)h_p(0) \neq 0$ we give solutions for special cases of quadratic systems. The assumption $h_p(0) \neq 0$ can be removed.

2 IDENTIFICATION

Identification is carried out in 3 special cases. The first case results when the terms of the diagonal $h_0(k), 0 \leq k \leq p$, are all equal. The second case results when all diagonal terms except $h_0(0), h_0(p)$ are zero. Finally the third case requires that $h_0(k), 1 \leq k \leq p-1$ alternate between zero and nonzero values.

2.1 The Diagonal Has Equal Terms

In this case eq.(2) implies

$$\gamma_{2,u}h_0(k) = \frac{c_{1,y}}{p+1} \qquad 0 \leq k \leq p \qquad (11)$$

Solving the linear system (10) we find the non diagonal terms scaled by the factor $\gamma_{3,u}^2/\gamma_{2,u}^2$. This is because the entries of H are formed by products of the form

$$h_0(k)h_0(l) = \frac{(c_{1,y})^2}{\gamma_{2,u}^2(p+1)^2}$$

To conclude a relation between the above factor and $\gamma_{2,u}$ must be found. This will ensure that all nondiagonal entries of the kernel are scaled by the same constant $\gamma_{2,u}$, as the diagonal entries do. Notice that

$$c_{4,y}(3p,2p,p) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(p)h_p(0)h_p(0)h_0(0) \qquad (12)$$

$$c_{3,y}(2p,p) = 2\gamma_{3,u}^2 h_0(p)h_p(0)h_0(0) \qquad (13)$$

Raising eq. (13) to the power of 2 and dividing it with eq. (12) we obtain

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u}} h_0(p) h_0(0) = \frac{(c_{3,y}(2p,p))^2}{c_{4,y}(3p,2p,p)}$$

Taking into account (11) we find

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u}} \frac{(c_{1,y})^2}{\gamma_{2,u}^2 (p+1)^2} = \frac{(c_{3,y}(2p,p))^2}{c_{4,y}(3p,2p,p)}$$

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u}^2} = \gamma_{2,u}(p+1)^2 \frac{(c_{3,y}(2p,p))^2}{c_{4,y}(3p,2p,p)(c_{1,y})^2}$$

2.2 Diagonal Terms Are Zero Except $h_0(0)$, $h_0(p)$.

If eq.(3) is evaluated for $i = p$ we have

$$\gamma_{2,u^2} h_0(0) h_0(p) = c_{2,y}(p) \tag{14}$$

The matrix H in eq. (10) is diagonal with diagonal entries $c_{2,y}(p)/\gamma_{2,u^2}$. Solving eq. (10) we determine the nondiagonal terms. As before caution is needed to ensure the same scaling factor between diagonal and nondiagonal terms. For this purpose we must find a relation between $\gamma_{3,u}^2/\gamma_{2,u^2}$ and $\gamma_{2,u}$. Using eqs. (12), (13), (14) we obtain

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u^2}} 2h_p(0) = \frac{c_{3,y}(2p,p)}{c_{2,y}(p)}$$

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u^2}} \frac{c_{4,y}(3p,2p,p)}{\gamma_{2,u} c_{3,y}(2p,p)} = \frac{c_{3,y}(2p,p)}{c_{2,y}(p)}$$

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u^2}} = \gamma_{2,u} \frac{(c_{3,y}(2p,p))^2}{c_{4,y}(3p,2p,p) c_{2,y}(p)}$$

Thus all nondiagonal terms are scaled by $\gamma_{2,u}$. Next we need to express γ_{2,u^2} in terms of $\gamma_{2,u}^2$. Using eqs. (3) and (2) we obtain

$$c_{2,y}(0) = \gamma_{2,u^2}(h_0^2(0) + h_0^2(p)) + 4\gamma_{2,u}^2 \sum_{k>0,n} h_k^2(n)$$

$$c_{2,y}(0) = \gamma_{2,u^2}\left(\frac{(c_{1,y})^2}{\gamma_{2,u}^2} - 2h_0(0)h_0(p)\right) + 4\gamma_{2,u}^2 \sum_{k>0,n} h_k^2(n)$$

$$\gamma_{2,u^2} = \gamma_{2,u}^2 \frac{c_{2,y}(0) + 2c_{2,y}(p) - 4\gamma_{2,u}^2 \sum_{k>0,n} h_k^2(n)}{(c_{1,y})^2}$$

we assume that $c_{1,y} \neq 0$ or $h_0(0) \neq -h_0(p)$. It follows from (2), (14) and the last relation that $\gamma_{2,u} h_0(0)$ and $\gamma_{2,u} h_0(p)$ satisfy the equations

$$\gamma_{2,u} h_0(0) + \gamma_{2,u} h_0(p) = c_{1,y}$$

$$\gamma_{2,u}^2 h_0(0) h_0(p) =$$

$$= \frac{(c_{1,y})^2}{c_{2,y}(0) + 2c_{2,y}(p) - 4\gamma_{2,u}^2 \sum_{k>0,n} h_k^2(n)} c_{2,y}(p)$$

Therefore the above terms are determined as the roots of a quadratic polynomial. Apprently an uncertaintly remains as to which of the roots correspond to the true parameter.

2.3 Diagonal Terms Alternate Between Zero And Nonzero Values

The procedure is best illustrated by an example. Let us assume that $p = 3$, and $h_0(1) = 0$. Then

$$c_{2,y}(3) = \gamma_{2,u^2} h_0(3) h_0(0) \tag{15}$$

$$c_{3,y}(6,3) = 2\gamma_{3,u}^2 h_0(3) h(3,0) h_0(0) \tag{16}$$

$$c_{4,y}(9,6,3) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(3,0) h(3,0) h_0(0) \tag{17}$$

Therefore

$$2\gamma_{2,u} h(3,0) = \frac{c_{4,y}(9,6,3)}{c_{3,y}(6,3)}$$

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u}} h_0(0) h_0(3) = \frac{(c_{3,y}(6,3))^2}{c_{4,y}(9,6,3)} \tag{18}$$

We compute the following $4-th$ order cumulants

$$c_{4,y}(8,6,3) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(2) h(3,0) h(3,0) h_0(0) +$$
$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(3,1) h(3,0) h_0(0)$$
$$c_{4,y}(8,5,3) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(2,0) h(3,0) h_0(0) +$$
$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(3,0) h(3,1) h_0(0)$$
$$c_{4,y}(8,5,2) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(3,0) h(2,0) h_0(0)$$

Simple algebraic manipulation lead to

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u}} h_0(0) h_0(2) =$$

$$\frac{c_{4,y}(8,6,3) + c_{4,y}(8,5,2) - c_{4,y}(8,5,3)}{4\gamma_{2,u}^2 h(3,0) h(3,0)} \tag{19}$$

$$2\gamma_{2,u} h(2,0) = \frac{c_{4,y}(8,5,2)}{c_{3,y}(6,3)}$$

$$2\gamma_{2,u} h(3,1) = \frac{c_{4,y}(8,5,3) - c_{4,y}(8,5,2)}{c_{3,y}(6,3)}$$

To determine the remaining nondiagonal terms we compute the following $4-th$ order cumulants

$$c_{4,y}(7,6,3) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(2) h(3,1) h(3,0) h_0(0) +$$
$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(3,2) h(3,0) h_0(0)$$
$$c_{4,y}(7,5,3) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(2) h(2,0) h(3,0) h_0(0) +$$
$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(2) h(3,0) h(3,1) h_0(0) +$$
$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(2,1) h(3,0) h_0(0) +$$
$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(3,1) h(3,1) h_0(0)$$
$$c_{4,y}(7,4,3) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(3) h(1,0) h(3,0) h_0(0) +$$

$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(3)h(2,0)h(3,1)h_0(0)+$$

$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(3)h(3,0)h(3,2)h_0(0)$$

$$c_{4,y}(7,4,1) = 4\gamma_{2,u}\gamma_{3,u}^2 h_0(3)h(3,0)h(1,0)h_0(0)+$$

$$4\gamma_{2,u}\gamma_{3,u}^2 h_0(3)h(3,0)h(3,0)h_0(2)$$

After some algebra we find

$$2\gamma_{2,u}h(3,2) =$$

$$= \frac{c_{4,y}(7,6,3) - \frac{\gamma_{3,u}^2}{\gamma_{2,u}}h_0(0)h_0(2)4\gamma_{2,u}^2 h(3,0)h(3,1)}{c_{3,y}(6,3)}$$

$$2\gamma_{2,u}h(2,1) = \frac{A}{c_{3,y}(6,3)}$$

where

$$A = c_{4,y}(7,5,3) - \frac{\gamma_{3,u}^2}{\gamma_{2,u}}h_0(0)h_0(2)4\gamma_{2,u}^2 h(3,0)h(2,0)-$$

$$-\frac{\gamma_{3,u}^2}{\gamma_{2,u}}h_0(0)h_0(2)4\gamma_{2,u}^2 h(3,0)h(3,1))-$$

$$-\frac{\gamma_{3,u}^2}{\gamma_{2,u}}h_0(0)h_0(3)4\gamma_{2,u}^2 h(3,1)h(3,1)$$

$$2\gamma_{2,u}h(1,0) = \frac{B}{c_{3,y}(6,3)}$$

and

$$B = c_{4,y}(7,4,3) - c_{3,y}(6,3)2\gamma_{2,u}h(3,2)-$$

$$-\frac{\gamma_{3,u}^2}{\gamma_{2,u}}h_0(0)h_0(3)4\gamma_{2,u}^2 h(2,0)h(3,1)$$

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u}}h_0(2)h_0(3) = \frac{c_{4,y}(7,4,1) - c_{3,y}(6,3)2\gamma_{2,u}h(1,0)}{4\gamma_{2,u}^2 h(3,0)h(3,0)}$$

So far we have found all the nondiagonal terms scaled by $\gamma_{2,u}$, and all products of terms on the main diagonal scaled by $\gamma_{3,u}^2/\gamma_{2,u}$. As before a relation between this factor and $\gamma_{2,u}^2$ must be determined. First γ_{2,u^2} is expressed in terms of $\gamma_{2,u}^2$. This is done via second order cumulants. Indeed

$$c_{2,y}(0) = \gamma_{2,u^2}(h_0^2(0)+h_0^2(2)+h_0^2(3))+4\gamma_{2,u}^2\sum_{k>0,n} h_k^2(n)$$

$$c_{2,y}(0) = \gamma_{2,u^2}(\frac{(c_{1,y})^2}{\gamma_{2,u}^2} - 2h_0(0)h_0(2) - 2h_0(0)h_0(3)-$$

$$-2h_0(2)h_0(3)) + 4\gamma_{2,u}^2\sum_{k>0,n} h_k^2(n)$$

$$c_{2,y}(0) = \frac{\gamma_{2,u^2}}{\gamma_{2,u}^2}(c_{1,y})^2 - 2c_{2,y}(3) - 2c_{2,y}(2) - 2c_{2,y}(1)+$$

$$+8\gamma_{2,u}^2\sum_{k>0,n} h_k(n)h_k(n+2)+8\gamma_{2,u}^2\sum_{k>0,n} h_k(n)h_k(n+1)+$$

$$+4\gamma_{2,u}^2\sum_{k>0,n} h_k^2(n)$$

Therefore

$$\gamma_{2,u^2} = \gamma_{2,u}^2\frac{\Gamma}{(c_{1,y})^2} \qquad (20)$$

where

$$\Gamma = c_{2,y}(0) + 2c_{2,y}(3) + 2c_{2,y}(2) + 2c_{2,y}(1)-$$

$$-8\gamma_{2,u}^2\sum_{k>0,n} h_k(n)h_k(n+2)-8\gamma_{2,u}^2\sum_{k>0,n} h_k(n)h_k(n+1)-$$

$$-4\gamma_{2,u}^2\sum_{k>0,n} h_k^2(n)$$

Finally using eqs. (15), (16), (17) and (20) we obtain

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u}\gamma_{2,u^2}} = \frac{(c_{3,y}(6,3))^2}{c_{4,y}(9,6,3)c_{2,y}(3)}$$

$$\frac{\gamma_{3,u}^2}{\gamma_{2,u}} = \gamma_{2,u}^2\frac{\Gamma(c_{3,y}(6,3))^2}{(c_{1,y})^2 c_{4,y}(9,6,3)c_{2,y}(3)} \qquad (21)$$

It follows from (2) that $\gamma_{2,u}h_0(0)$, $\gamma_{2,u}h_0(2)$ and $\gamma_{2,u}h_0(3)$ satisfy the equation

$$\gamma_{2,u}h_0(0) + \gamma_{2,u}h_0(2) + \gamma_{2,u}h_0(3) = c_{1,y}$$

Using eqs. (18), (19) and (21) we obtain

$$\gamma_{2,u}^2 h_0(0)(h_0(2) + h_0(3)) = K$$

where K is easily derived. Once again $\gamma_{2,u}h_0(0)$ is determined as a root of a quadratic polynomial.

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