WEIGHTED LAGRANGIAN INTERPOLATING FIR FILTER

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ABSTRACT *The aim of this paper is to present an algorithm for the coefficients of a weighted Lagrangian interpolating FIR filter. The proposed filter is effective in the reduction of amplitude response sidelobes responsible for aliasing. The novelty of the proposed L-th band interpolating filter lies in that it allows for a simultaneous L-fold interpolation and fractional sample delaying of an input signal. The filter can be recommended for on-line resampling in variable delay situations especially when implemented in the so-called modified Farrow structure.*

1. STATE OF THE ART

Digital interpolation filtering is a topic that arises great interest in digital signal processing literature. This is because there are many situations, e.g. in timing recovery for high speed data communications, adaptive delay tracking, delay control systems, precision beamforming and beamsteering etc., where there is a need for resampling an original discrete-time signal from one to another sample rate.

In those applications, where resampling must be realized on-line, one seeks for simple solutions convenient in practical hardware, offering rapid redesign. Then the techniques with closed-form design formulas for the filter coefficients are favoured. Such closed-form solutions have been hitherto derived for windowing, frequency sampling, least-square, Lagrangian [1],[2],[8],[9] and piecewise parabolic [3] FIR filter design methods. Among the above methods the Lagrangian and piecewise parabolic interpolation filter types are known as unsurpassed from the point of view of the smallest approximation error in the vicinity of the centre frequency of the filter frequency response. However, the main drawback of the Lagrangian and piecewise parabolic Lth band filters used for interpolation by a factor of L, is the presence of relatively high amplitude response sidelobes in the so-called ϕ bands [4] centred at the frequencies $\omega_r = (2r+1)\pi/L$, $r = 1,2,\ldots$. As a result, these filters do not guarantee that the tails of the input signal spectrum, as well as any noise that normally exists in these bands, are not amplified so much that they become excessive in the output signal.

In this paper we propose a technique aimed at reducing the sidelobe level of the Lagrangian Lth band filters.

2. DEFINITIONS

An Lth band filter is understood here as a lowpass filter whose ideal frequency response is defined as

$$H_{\varepsilon L}(e^{j\omega}) \triangleq \begin{cases} \exp(-j\omega\varepsilon), & |\omega| < \pi/L \\ 0, & \pi/L < |\omega| < \pi \end{cases} \quad (1)$$

where ε in Sa stands for a fractional delay introduced by the filter and Sa is the symbol of sample interval. Consequently, the frequency response of an ideal one-band filter of delay ε, thus a filter with L=1, is

$$H_{\varepsilon}(e^{j\omega}) \triangleq \exp(-j\omega\varepsilon), \quad |\omega| < \pi \quad (2)$$

This filter is known under the name of fractional sample delay (FSD) filter. Both ideal filters defined above are noncausal, thus nonrealizable in real time in an on-line manner.

The impulse response of a causal N-point Lagrangian FIR FSD filter is given by (cf. [2],[5])

$$h_{\varepsilon N}[n] \triangleq \prod_{\substack{\nu=0 \\ \nu\neq n}}^{N-1} \frac{\varepsilon + (N-1)/2 - \nu}{n - \nu} \quad (3)$$

The delay $\tau \triangleq \varepsilon + (N-1)/2$ Sa introduced by this filter is generally noninteger. The frequency response $H_{\varepsilon N}(e^{j\omega})$ of this filter fulfils the following maximum flatness conditions

$$d^k E_{\varepsilon N}(e^{j\omega}) / d\omega^k = 0, \quad \omega = 0, \quad k = 0,1,\ldots,N-1 \quad (4)$$

where $E_{\varepsilon N}(e^{j\omega}) \triangleq H_{\varepsilon}(e^{j\omega}) - H_{\varepsilon N}(e^{j\omega})$ is the complex frequency response approximation error. The magnitude of this error is the smallest for $|\varepsilon| < 0.5$ [2],[5].

The impulse response $h_{\varepsilon LN}[n], n = 0,1,..,NL-1$ of a Lagrangian Lth band filter of length LN can be obtained by polyphase composition (the technique inverse to polyphase decomposition [6]) of L Lagrangian FSD subfilters: $h_{\varepsilon_0,N}[n]$, $h_{\varepsilon_1,N}[n]$, ... , $h_{\varepsilon_{L-1},N}[n]$, $n = 0,1,\ldots,N-1$ with $\varepsilon_l = \varepsilon + l/L$, $l = 0,1,\ldots,L-1$. The delay introduced by the lth subfilter is $\varepsilon_l + (N-1)/2$. If the polyphase composition is realized in such a manner that the impulse response of the Lth band FIR filter is related to the above impulse responses of its FSD subfilters as given by

$$h_{\varepsilon LN}[nL + l] = h_{\varepsilon_{L-1-l},N}[n], \qquad l = 0,1,..,L-1, \quad n = 0,1,\ldots,N-1 \quad (5)$$

then the delay introduced by the Lth band filter is $\varepsilon L + (L-1)/2 + (NL-1)/2$ Sa.

3. NEW CONTRIBUTION

The idea of our weighted Lagrangian FIR filter is based on using for the FSD filter (or subfilter) impulse response the following linear combination of the impulse responses of a pair of Lagrangian FIR FSDs of length N each but of delays differing by one Sa

$$h^{(w)}_{\varepsilon,N+1}[n] \triangleq \begin{cases} (1-\varepsilon)h_{\varepsilon N}[0], & n = 0 \\ (1-\varepsilon)h_{\varepsilon N}[n] + \varepsilon h_{\varepsilon-1,N}[n-1], & n = 1,\ldots,N-1 \\ \varepsilon h_{\varepsilon-1,N}[N-1], & n = N \end{cases} \quad (6)$$

The transfer function of this filter is

$$H^{(w)}_{\varepsilon,N+1}(z) = (1-\varepsilon)H_{\varepsilon,N}(z) + \varepsilon z^{-1} H_{\varepsilon-1,N}(z) \quad (7)$$

where $H_{\varepsilon,N}(z)$ and $H_{\varepsilon-1,N}(z)$ stand for the transfer functions of the filters $\{h_{\varepsilon,N}[n]\}_0^{N-1}$ and $\{h_{\varepsilon-1,N}[n]\}_0^{N-1}$ respectively. We have shown that the frequency response $H^{(w)}_{\varepsilon,N+1}(e^{j\omega})$ of the weighted Lagrangian FIR FSD filter (6) fulfils the following maximum flatness conditions

$$d^k E^{(w)}_{\varepsilon,N+1}(e^{j\omega}) / d\omega^k = 0, \quad \omega = 0, \quad k = 0,1,\ldots,N-1$$

where all the derivatives of the complex frequency response approximation error

$$E^{(w)}_{\varepsilon,N+1}(e^{j\omega}) \stackrel{\Delta}{=} H_\varepsilon(e^{j\omega}) - H^{(w)}_{\varepsilon,N+1}(e^{j\omega})$$

up to the N-1 derivative for this N+1-point FIR filter are of zero value.

Each of the component Lagrangian filters produce an estimate of the desired output sample $x(n-\tau) \stackrel{\Delta}{=} x(t)\big|_{t=(n-\tau)T}$ shifted in time relative to the current input signal sample $x[n] \stackrel{\Delta}{=} x(t)\big|_{t=nT}$, where $\tau \stackrel{\Delta}{=} \varepsilon + (N-1)/2$ Sa as in Sect.2. The target estimate $\hat{x}(n-\tau)$ of $x(n-\tau)$ is obtained by weighting these two estimates by $1-\varepsilon$ and ε respectively.

In a number of applications the problem is to synthesize a very short interpolating FIR filter offering rapid redesign, suitable for realization of irrational resampling rates. The need for such a filter is presently found especially in digital speech and audio processing. The examples of sample rates to be mutually converted are: telephone PCM - 8000 Sa/s, CD - 44100 Sa/s, DAT - 48000 Sa/s, DCC/DAB/DAR - 32000 Sa/s. Other possible applications are listed in Sect.1. Consequently, we shall confine now to the third order weighted Lagrangian FIR FSD filter which will be exploited further as the basic building block for a weighted Lagrangian Lth band filter. The order N=3 can be considered as the smallest for reasonable accuracy of approximation.

The third order weighted Lagrangian FSD filter is composed of the following pair of Lagrangian FSD subfilters of length N=3, fractional delay ε and total delay τ

$$H_{\varepsilon,3}(z) = h_{\varepsilon,3}[0] + h_{\varepsilon,3}[1]z^{-1} + h_{\varepsilon,3}[2]z^{-2} \qquad (8)$$

and

$$H_{\varepsilon-1,3}(z) = h_{\varepsilon-1,3}[0] + h_{\varepsilon-1,3}[1]z^{-1} + h_{\varepsilon-1,3}[2]z^{-2} \qquad (9)$$

where

$$h_{\varepsilon,3}[0] = \varepsilon(\varepsilon-1)/2, \quad h_{\varepsilon,3}[1] = 1-\varepsilon^2, \quad h_{\varepsilon,3}[2] = \varepsilon(\varepsilon+1)/2 \quad (10)$$

and

$$h_{\varepsilon-1,3}[0] = (\varepsilon-1)(\varepsilon-2)/2, h_{\varepsilon-1,3}[1] = \varepsilon(2-\varepsilon), h_{\varepsilon-1,3}[2] = \varepsilon(\varepsilon-1)/2 \qquad (11)$$

The transfer function of a weighted Lagrangian FSD filter of length N+1=4, fractional delay ε and total delay τ as above is

$$H^{(w)}_{\varepsilon,4}(z) = h^{(w)}_{\varepsilon,4}[0] + h^{(w)}_{\varepsilon,4}[1]z^{-1} + h^{(w)}_{\varepsilon,4}[2]z^{-2} + h^{(w)}_{\varepsilon,4}[3]z^{-3} \qquad (12)$$

where, according to (6),

$$h^{(w)}_{\varepsilon,4}[0] = -\varepsilon(1-\varepsilon)^2/2, \quad h^{(w)}_{\varepsilon,4}[1] = (2-5\varepsilon^2+3\varepsilon^3)/2,$$
$$h^{(w)}_{\varepsilon,4}[2] = \varepsilon(1+4\varepsilon-3\varepsilon^2)/2, \quad h^{(w)}_{\varepsilon,4}[3] = \varepsilon^2(\varepsilon-1)/2 \qquad (13)$$

By substituting (13) to (12) and rearranging the resulting polynomial in fractional delay ε (instead of z^{-1}) we obtain the following representation of weighted Lagrangian filter transfer function

$$H^{(w)}_{\varepsilon,4}(z) = C_0(z) + C_1(z)\varepsilon + C_2(z)\varepsilon^2 + C_3(z)\varepsilon^3 \qquad (14)$$

where $C_n(z)$, $n = 0,1,2,3$ are polynomials in z^{-1} as given by

$$C_0(z) = z^{-1}, C_1(z) = -1/2 + z^{-2}/2, C_2(z) = 1 - 5z^{-1}/2 + 2z^{-2} - z^{-3}/2,$$

$$C_3(z) = -1/2 + 3z^{-1}/2 - 3z^{-2}/2 + z^{-3}/2 \qquad (15)$$

This representation leads directly to the modified Farrow structure [7],[8] of our third-order weighted Lagrangian FSD filter, shown in Fig.1. The transfer functions $C_n(z)$ sharing the unit delays are fixed for given order N=3 and in Fig.1 the only parameter to be changed is ε. The modification of the original Farrow structure is based here on the Välimäki strategy [7] that the varying parameter is ε rather than τ (generally noninteger). This makes the Farrow structure especially efficient for applications where the fractional delay ε is changed often, even every sample interval.

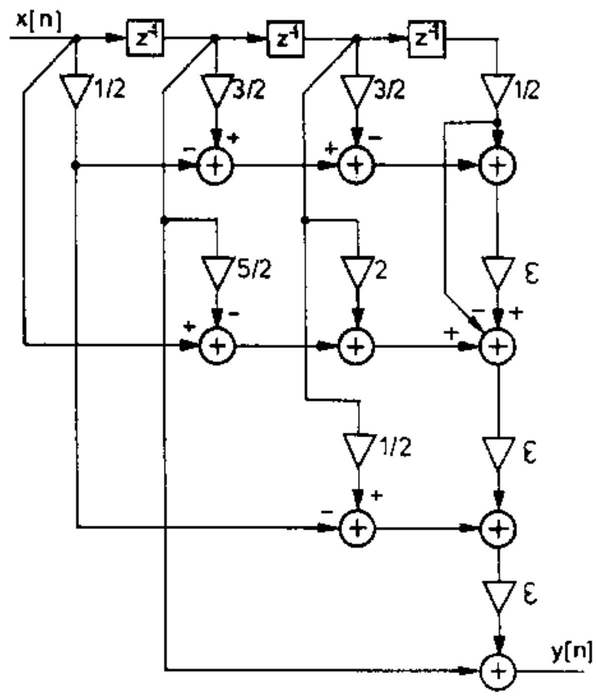

Fig.1. The modified Farrow structure for a third order (N=3) weighted Lagrange interpolator

It is worth noting that for fractional $\varepsilon \neq 0$ an Lth band filter composed of weighted Lagrangian FSD subfilters acts as an L-fold interpolator and resampler simultaneously, while for $\varepsilon = 0$ it resolves to the standard linear-phase L-fold interpolator [4],[6] which fulfils the Nyquist condition [6] for leaving the interpolation nodes, thus the input signal samples in the interpolator output signal, unchanged.

4. PERFORMANCE

Typical curves for the assessment of the performance of our Lth band weighted Lagrangian FIR filter composed of third order weighted Lagrangian FSD subfilters are presented in Fig.2. It shows the amplitude response and the net group delay response of the weighted Lagrangian (dotted line), Lagrangian and piecewise parabolic Lth band filters (solid lines) with the parameters: $L = 3, N+1 = 4, \varepsilon = 0.25$ Sa. For the sake of comparison the net group delay responses of the Lagrangian and piecewise parabolic filters are in Fig.1 turned vertically around ε. The location of transfer function zeros for these three filters is depicted in Fig.3 with o-mark, x-mark and +-mark, respectively. The coefficients of the filters are:

$$\{h^{(w)}_{\varepsilon,L,N+1}[n]\}_0^{(N+1)L-1}\Big|_{\substack{N=L=3\\ \varepsilon=0.25}} = \{-11,-175,-243,189,$$

$$1545,2997,3399,2331,783,-121,-245,-81\}/10368$$

for a weighted Lagrangian filter,

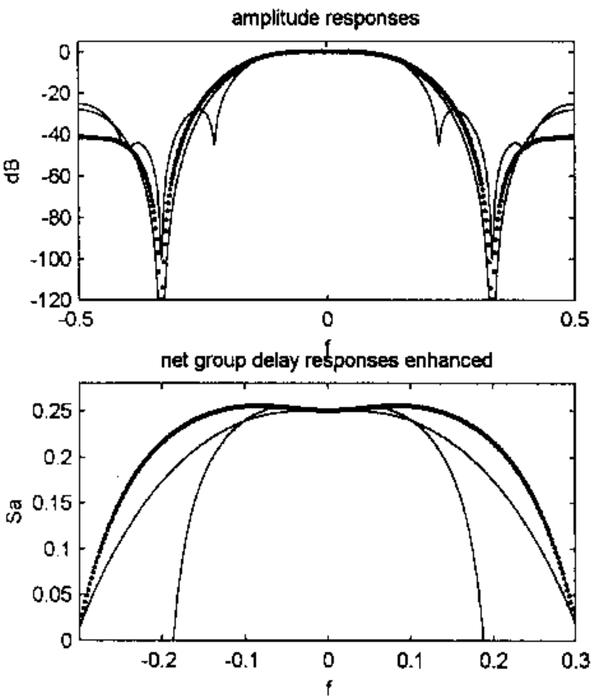

Fig.2. The amplitude responses and the net group delay responses of weighted Lagrangian (dotted line), Lagrangian and piecewise parabolic (solid lines) Lth band filters with the parameters: $L=3, N+1=4, \varepsilon=0.25$ Sa (the inmost curves are for the piecewise parabolic filter)

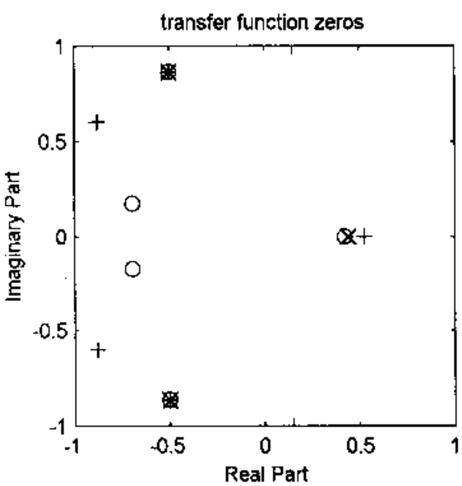

Fig.3. The location of transfer function zeros for weighted Lagrangian (o-mark), Lagrangian (x-mark) and piecewise parabolic (+-mark) Lth band FIR filter with the parameters as in Fig.1 (zeros far from the unit circle are not shown here)

$$\{h_{\varepsilon-0.5,L,N+1}[n]\}_0^{(N+1)L-1}\Big|_{\substack{N=L=3\\ \varepsilon=0.25}} = \{-143,-595,-567,897,$$

$$4845,8505,9867,6783,2835,-253,-665,-405\}/31104$$

for a standard Lagrangian filter [2,5] of the same total delay and length and

$$\{-11,-35,-27,35,155,243,275,203,99,-11,-35,-27\}/864$$

for a piecewise parabolic filter. The coefficients of the piecewise parabolic filter have been computed using the two-parameter algorithm given in [3] with the parameters: μ-the fractional delay and α-the weight. Here we have used $\mu=\varepsilon$. The second parameter has been set to $\alpha=0.5$ for simple hardware implementation (see p. 1004 and Appendix in [3]).

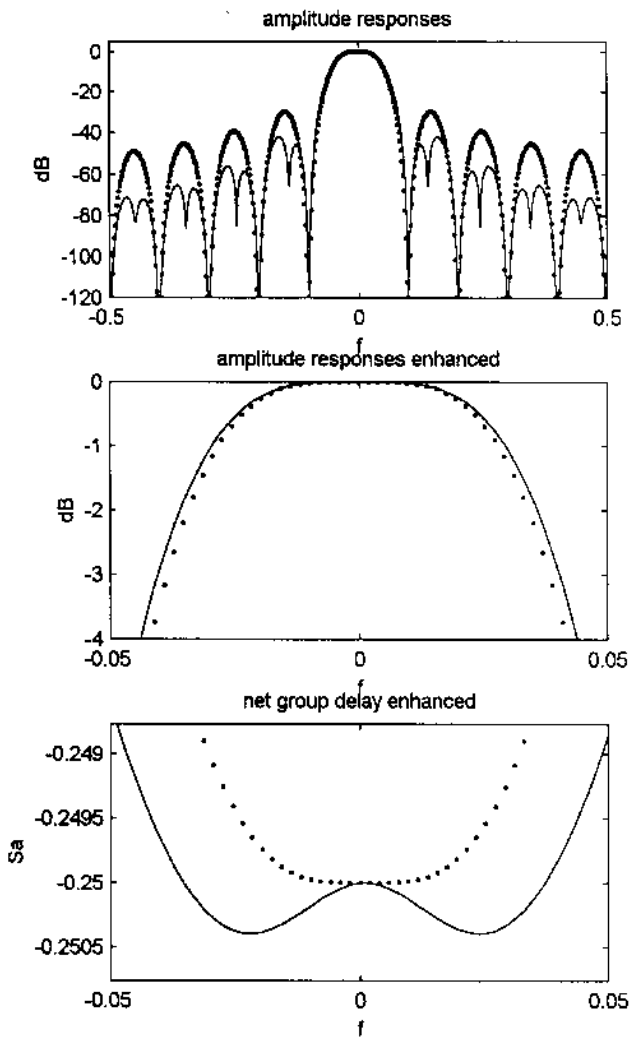

Fig.4. The total amplitude responses in dB, the amplitude responses enhanced in the passband and the net group delay responses in the passband of weighted Lagrangian (solid line) and Lagrangian (dotted line) Lth band filters with the parameters: $L=10$, $N=3$ and fractional delay 0.25 Sa

Figs.4 and 5 show the effect of increasing the interpolation factor L (from L=3 as in Fig.2 to L=10) on the amplitude and group delay responses of weighted Lagrangian Lth band filter (solid line) in comparison with the Lagrangian filter (dotted line in Fig.4) and the piecewise parabolic filter (dotted line in Fig.5), with the remaining parameters: the length of the subfilters and the fractional delay unchanged. We have shown the total amplitude responses in dB for $f = \omega / 2\pi \in \langle -0.5, 0.5)$ 1/Sa and the amplitude and the group delay responses of these filters enhanced in the passband. Only the fractional part of the group delay responses is shown, approximating the desired constant fixed here to 0.25 Sa. The ripples of the group delay response exhibited by the weighted Lagrangian and piecewise parabolic filters in he passband are in fact very small and do not exceed 1% of the desired constant value.

These examples, as well as other experiments with a wide set of the parameters: L, N and ε, show that:

- The magnitude of weighted Lagrangian filter amplitude response sidelobes is significantly reduced (from approximately -10 dB to more than -30 dB depending mainly on N and L) as compared with its Lagrangian or piecewise parabolic counterparts.
- Weighting the Lagrangian Lth band filter coefficients as in Sect.3, locates the transfer function zeros, responsible for the sidelobe reduction, quite close to the unit circle, at the angles corresponding to angular frequencies very close to the frequencies ω_r from Sect.1.

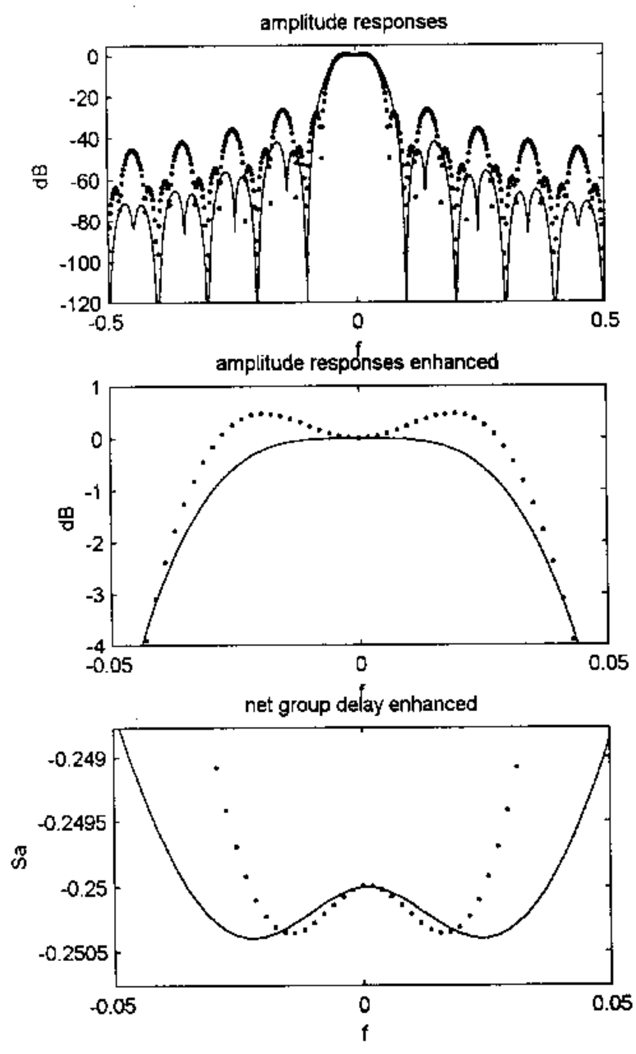

Fig.5. The total amplitude responses in dB, the amplitude responses enhanced in the passband and the net group delay responses in the passband of weighted Lagrangian (solid line) and piecewise parabolic (dotted line) Lth band filters with the parameters: $L = 10$, $N = 3$ and fractional delay 0.25 Sa

- The amplitude response and the group delay response bandwidths with good approximation are slightly wider for the weighted Lagrangian than for a nonweighted Lagrangian filter. For a piecewise parabolic filter both these bandwidths are the smallest.
- The coefficients of all three filter types under consideration can be realized exactly in fixed-point arithmetic with simplest hardware for piecewise parabolic filter and with hardware which is simpler for a weighted Lagrangian than for a nonweighted Lagrangian interpolator.

5. CONCLUSIONS

The original contribution of this paper is an algorithm for the coefficients of a weighted Lagrangian interpolating FIR filter. It is shown that this algorithm can be implemented in a modified Farrow structure well suited to the applications where the coefficients are updated often, even every sample interval. The performance of this filter is improved as compared with the nonweighted Lagrangian filter and piecewise parabolic filter. The proposed solution is effective in the reduction of amplitude response sidelobes responsible for aliasing. The novelty of the proposed L-th band interpolating filter lies in that it allows for a simultaneous L-fold interpolation and fractional sample delaying of an input signal. The filter can be recommended for on-line resampling and delay compensation in variable delay situations.

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